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D-COMPLETE AXIOMS FOR THE CLASSICAL EQUIVALENTIAL CALCULUS

Abstract

Virtually all previously known axiom sets for the classical equivalential calculus, \( \text{EQ} \), are \( \text{D} \)-incomplete: not all theorems derivable by substitution and detachment can be derived using the rule \( \text{D} \) of condensed detachment alone. The only exception known to the author is Wajsberg’s base \( \{ \text{EE}p\text{E}qr\text{E}qp, \text{EE}p\text{ppp} \} \). This axiom set, albeit inorganic since one of its members contains a theorem of \( \text{EQ} \) as a proper subformula, is here shown to be \( \text{D} \)-complete. \( \text{D} \)-complete single axioms for \( \text{EQ} \) are then constructed, culminating with \( \text{EE}pq\text{EE}rq\text{EsEsEsEsEpr} \) which has the distinction of being both \( \text{D} \)-complete and organic.

1. \( \text{D} \)-completeness and \( \text{D} \)-incompleteness

The well-formed formulas of the classical equivalential calculus are built in the usual way from a binary connective \( E \) and denumerably many sentence letters \( p, q, r, \ldots, p_1, p_2, \ldots \): each sentence letter is well-formed, and if \( \alpha \) and \( \beta \) are well-formed, so is \( E\alpha\beta \).

Let \( \text{EQ} \) be the set of such formulas in which each sentence letter occurring occurs an even number of times. The members of \( \text{EQ} \) are, as first noted by Leśniewski [7], exactly the formulas that are tautologies of the standard two-valued truth-table for material equivalence.

For each set \( A \) of formulas, let the \( A \)-theorems be the formulas derivable from members of \( A \) using the rules of substitution and detachment (that is, \textit{modus ponens}). A subset \( A \) of \( \text{EQ} \) is a \textit{complete axiom set}, or \textit{base}, for \( \text{EQ} \), if and only if the set of \( A \)-theorems is exactly \( \text{EQ} \).
C. A. Meredith’s rule **D** of condensed detachment, which combines detachment with a small amount of substitution, has been a tool of choice in the investigation of sentential calculi such as **EQ** since its introduction in [14]. Detailed presentations of rule **D** may be found in, for example, [2], [4], and [6]. Briefly, let $E\alpha\beta$ (the major premise) and $\gamma$ (the minor premise) be any pair of formulas. Let $\gamma'$ be an alphabetic variant of $\gamma$ containing no letters occurring in $E\alpha\beta$. If there exists a substitution $s$ of formulas for the letters occurring in $\alpha$ and $\gamma'$ for which $s(\alpha) = s(\gamma')$, $\alpha$ and $\gamma'$ are said to be **unifiable**, and $s$ is called a **unifier** for them. When $\alpha$ and $\gamma'$ are unifiable, there is, by the Unification Theorem of [15], always a most general unifier for the two, that is, a substitution $s$ of formulas for the letters occurring in $\alpha$ and $\gamma'$ such that for any other substitution $s'$ unifying $\alpha$ and $\gamma'$, $s'(\alpha)$ is a substitution instance of $s(\alpha)$. By trivial letter-for-letter alterations, $s$ can always be turned into a $\beta$-acceptable most general unifier $s^*$ for $\alpha$ and $\gamma'$, that is, a most general unifier for $\alpha$ and $\gamma'$ for which no sentence letters occurring in $s^*(\alpha)$ occur in $\beta$ but not $\alpha$. Rule **D** can be formulated as follows:

**Rule D. Premises:** Any two formulas $E\alpha\beta$ and $\gamma$ for which $\alpha$ and $\gamma$ have a common substitution instance. **Conclusion:** any alphabetic variant of $s^*(\beta)$ where $\gamma'$ is any alphabetic variant of $\gamma$ containing no sentence letters in common with $E\alpha\beta$ and $s^*$ is any $\beta$-acceptable most general unifier for $\alpha$ and $\gamma'$.

Following Hindley and D. Meredith [4], for each set $\mathbf{A}$ of formulas, let the **condensed** $\mathbf{A}$-**theorems** be the formulas **D**-derivable from those in $\mathbf{A}$, that is, those derivable by **D** alone. Clearly every condensed $\mathbf{A}$-theorem is an $\mathbf{A}$-theorem, and though the converse is not true in general, we do have in its place a well known and fundamental lemma:

**Lemma 1.** (Kalman [6]) For each set $\mathbf{A}$ of formulas, every $\mathbf{A}$-theorem is a substitution instance of at least one condensed $\mathbf{A}$-theorem.

If every $\mathbf{A}$-theorem is a condensed $\mathbf{A}$-theorem—that is, if every formula deducible from $\mathbf{A}$ by substitution and detachment is also deducible by $\mathbf{D}$ alone—$\mathbf{A}$ is (cf. [12]) **D**-**complete**. Otherwise, $\mathbf{A}$ is **D**-**incomplete**.

The first $\mathbf{D}$-completeness and $\mathbf{D}$-incompleteness results were given by Hindley and D. Meredith [4], who employ the binary connective $\mathbf{C}$ rather than $\mathbf{E}$. With $\mathbf{B} = CCpqCCrpCrq$, $\mathbf{C} = CCpqCqrCqp$, $\mathbf{K} = CpCqp$, $\mathbf{W}$
= CCpCpqCpq, and I = Cpp, they showed that the axiom sets \{B, C, K, CCCpqpp\} and \{B, C, K, W\} for classical and intuitionistic implication, respectively-and, indeed, all extensions of the latter-are D-complete, while Meredith's \cite{11} sets \{B, C, I\} and \{B, C, K\} are D-incomplete. Meyer and Bunder \cite{12} subsequently established D-completeness for all extensions of the bases \{B, CCpqCCqrCpr, I, W\} and (cf. Mints and Tammet \cite{13}) \{B, C, I, W\} for the relevance logics T_\rightarrow and R_\rightarrow. More recently, the latter results have been extended by Megill and Bunder \cite{10} to a family of still weaker axiom sets.

2. Axiom Sets for Classical Equivalence

Following Belnap \cite{1}, a formula has the two-property just in case each sentence letter occurring in it occurs exactly twice. It is a striking fact that virtually all previously reported axiom sets for EQ consist exclusively of axioms with the two-property, from Leśniewski's \cite{7} original axiom set \{EEpqErpEqr, EEpEqrEEpqr\} through numerous additional bases reported in \cite{8} and \cite{5} to the fourteen shortest possible single axioms from various hands listed in \cite{17} (and named as in \cite{3}):

\begin{align*}
\text{YQL. } EEpqEEqErp & \quad \text{YQF. } EEpqEEprEqr \\
\text{UM. } EEEpqrEEqErp & \quad \text{XGF. } EpEEqEprEqr \\
\text{YRM. } EEpqErEEqErp & \quad \text{YRO. } EEpEqrEEpqr \\
\text{PYM. } EEEpErqEEqErp & \quad \text{XGK. } EpEEqErpEqr \\
\text{XHN. } EpEEqEEpqr & \quad \text{XCB. } EpEEpEqErqr.
\end{align*}

By a classic result in \cite{1} (cf. \cite{2}), rule D preserves the two-property, that is, for each set A of formulas, if every member of A has the two-property then so does each condensed A-theorem. In the case of such bases for EQ, we can identify their condensed theorems precisely:

**Lemma 2.** If A is a complete axiom set for EQ and each member of A has the two-property, then the condensed A-theorems are exactly the formulas with the two-property.

**Proof.** Where A is any complete axiom set for EQ in which each member has the two-property, Belnap's preservation result ensures that every condensed A-theorem has the two-property.
For the converse, we first prove, by induction on length, that for each formula \( \alpha \) in which no sentence letter occurs more than once, \( E\alpha \alpha \) is a condensed \( \mathbf{A} \)-theorem. Where \( \alpha \) is a sentence letter, \( E\alpha \alpha \) is a condensed \( \mathbf{A} \)-theorem by Lemma 1. For the induction step, choose any formula \( \alpha = E\beta\gamma \) in which no letter occurs more than once and assume, on inductive hypothesis, that \( E\beta\beta \) and \( E\gamma\gamma \) are condensed \( \mathbf{A} \)-theorems. By Lemma 1 again, \( EEpqEErsEEprEqs \) is also a condensed \( \mathbf{A} \)-theorem. With that theorem as major premise and \( E\beta\beta \) as minor, \( \mathbf{D} \) gives \( EErsEE\beta\beta s \). With the latter theorem as major and \( E\gamma\gamma \) as minor, a second application of \( \mathbf{D} \) then gives \( EE\beta\gamma E\beta\gamma \), that is, \( E\alpha \alpha \).

Now consider any formula, \( \tau \), with the two-property. Then \( \tau \) is in \( \mathbf{EQ} \) and so, by Lemma 1, is a substitution instance of some condensed \( \mathbf{A} \)-theorem, \( \tau' \). Let \( \alpha \) be like \( \tau \) but with each occurrence of a sentence letter in \( \tau \) replaced with a distinct new letter. \( E\alpha \alpha \) is then a condensed \( \mathbf{A} \)-theorem, whence application of \( \mathbf{D} \) with \( E\alpha \alpha \) as major and \( \tau' \) as minor gives \( \tau \).

Most of the known bases for \( \mathbf{EQ} \), then, are \( \mathbf{D} \)-incomplete. In fact, the only base with an axiom lacking the two-property the author has found in the literature is Wajsberg’s [16] axiom set \( \{ EEpqEqr Erp Eqp, EEEpppp \} \). To show that this set is, indeed, \( \mathbf{D} \)-complete, we employ a lemma from [10]:

**Lemma 3.** (Megill-Bunder [10]) For each set \( \mathbf{A} \) of formulas, if \( EEpqEEqr Erp Eqp, EEpqEErp Eqp, Epp, \) and \( EEqpEpEqpq \) are condensed \( \mathbf{A} \)-theorems, then (i) each substitution instance \( E\alpha \alpha \) of \( Epp \) is a condensed \( \mathbf{A} \)-theorem and (ii) \( \mathbf{A} \) is therefore \( \mathbf{D} \)-complete.

**Proof.** Details are given in [10]. That (i) follows from the hypothesis of the Lemma is established inductively along considerably more intricate lines than those used in the proof of Lemma 2 above. That (i) in turn gives (ii) is shown to hold for any set \( \mathbf{A} \) of formulas whatever: each \( \mathbf{A} \)-theorem, \( \tau \), must be an instance of a condensed \( \mathbf{A} \)-theorem, \( \tau' \), whence application of \( \mathbf{D} \) with \( E\tau\tau \) as major and \( \tau' \) as minor delivers \( \tau \).

**Theorem 4.** \( \{ EEpqEqr Erp Eqp, EEEpppp \} \) is a \( \mathbf{D} \)-complete base for \( \mathbf{EQ} \).

**Proof.** Since Wajsberg’s base is complete for \( \mathbf{EQ} \), Lemma 2 assures that \( EEpqEEqr Erp, EEpqEErp Eqp, \) and \( Epp \) are among its condensed theorems. To complete the proof, then, it is enough, by Lemma 3, to show that
$EEpEpqEpEpq$ is also $D$-derivable. As is standard practice, let $D_{m.n}$ be a representative alphabetic variant of the result of applying rule $D$ to formula $m$ as major and formula $n$ as minor. We have:

1. $EEpEqErEqp$
2. $EEpppp$

$D1.1 = 3. EEpqEqErEqpr$
$D3.1 = 4. EpEEqErsEEsEpq$
$D1.4 = 5. EEEpEqrsEErEqps$
$D5.3 = 6. EEpqEqsEESpEqr$
$D1.6 = 7. EEpqErsEsEqEpr$
$D7.1 = 8. EEpqEEppErq$
$D7.6 = 9. EEpqErsEEprEqp$
$D8.9 = 10. EEEpqErsEEEqEqs$
$D9.8 = 11. EEpqEqErEqr$
$D10.2 = 12. EEEpqEppEppEpq$
$D12.10 = 13. EEppEpq$
$D13.11 = 14. EEpqEpEpq$

Though Wajsberg’s base is unique among previously reported axiom sets for $EQ$, other such bases—even single axioms—can be readily constructed. To convert the $D$-incomplete single axiom $YQF = EEpqEEprEqr$ to a $D$-complete single axiom, for example, it is enough to prefix $EsEsEsEs \ldots$ to it:

**Theorem 5.** \{EsEsEsEsEEpqEEprEqr\} is a $D$-complete base for $EQ$.

**Proof.** The single axiom $YQF$ is clearly $D$-derivable from this base:

1. $EsEsEsEsEEpqEEprEqr$
$D1.1 = 2. EEpEpEpEpEEqEEqsErsEEpEpEpEEqrEEqsErsEEpEpEpEEqrEEqsErsEE\ldots$
$D2.1 = 3. EEpEpEpEpEEqEEqsErsEEpEpEpEppEpEEqrEEqsErsEE\ldots$
$D3.1 = 4. EEpEpEpEpEEqEEqsErsEE\ldots$
$D4.1 = 5. EEpqEEprEqr$

By Lemma 2, Wajsberg’s first axiom, $EEpEqErEqp$, is then $D$-derivable as well. To complete the proof of $D$-completeness it therefore suffices, by Theorem 4, to show that his second, $EEpppp$, is also $D$-derivable:
This method of constructing single axioms can be generalized:

**Theorem 6.** Let \( \mathcal{A} \) be any single axiom for \( \text{EQ} \), with \( s \) any letter not occurring in \( \mathcal{A} \). Then \( \{EsEsEsEs\mathcal{A}\} \) is a \( \mathcal{D} \)-complete base for \( \text{EQ} \).

**Proof.** Four successive detachments deliver \( \mathcal{A} \), from which, by Lemma 2, \( EEpEqrErEqp \) is \( \mathcal{D} \)-derivable. On the same grounds, \( EEWzEyEz\mathcal{A}-EEEwxyz \) (for \( w, x, y, z \) not in \( \mathcal{A} \)) is \( \mathcal{D} \)-derivable. With the latter as major premise and \( EsEsEsEs\mathcal{A} \) as minor, \( EEEpppp \) then follows by a single application of \( \mathcal{D} \), ensuring \( \mathcal{D} \)-completeness by Theorem 4\(^1\).

The \( \mathcal{D} \)-complete single axioms for \( \text{EQ} \) thus constructed are, as is Wajsberg’s \( \mathcal{D} \)-complete two-base, *inorganic* in the sense that each includes, as a proper subformula, another theorem of \( \text{EQ}-\text{Epp} \) in the Wajsberg case, a known single axiom in the other cases-while the alternate bases given in the literature, though organic, are \( \mathcal{D} \)-incomplete. It is, however, possible to construct a base for \( \text{EQ} \)-indeed, a single axiom-which is both \( \mathcal{D} \)-complete and organic:

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\(^1\)If \( \mathcal{A} \) is one of the single axioms YQL, YQF, YQJ, UM, XGF, YRM, XGK, XHK, or XHN, \( EEEE\mathcal{A}ssss \) is also a \( \mathcal{D} \)-complete single axiom for \( \text{EQ} \). This construction lacks the generality of the method of Theorem 6 but is useful for systems other than \( \text{EQ} \). For example, \( CCCCCpCqrCCCCuvCllCllpssss \) is a \( \mathcal{D} \)-complete single axiom for \( \text{BCI} \), as is \( CCCCCCCCCpCqrCCCCuvCllCllpssss \) for \( \text{BCK} \).
Theorem 7. \{EEpqEE rqEsEsEsEpr\} is an organic D-complete base for EQ.

Proof. The formula in question is clearly organic since none of its proper subformulas have the two-property. Its D-completeness is assured by the following derivation from it of the D-complete axiom of Theorem 5:

1. \(EEpqEE rqEsEsEsEpr\)
   D1.1 = 2. \(EEpEE qrEsEsEsE tqE uEuEu Eu E trp\)
   D2.1 = 3. \(EpEpEpEpEE qrEqr\)
   D3.3 = 4. \(EEpEpEpEpEE qrEqrEEpEpEpEE qrEqrEE pr-EEqrEqrEEstE est\)
   D4.3 = 5. \(EEpEpEpEpEE qrEqrEEpEpEpEE qrEqrEEstE est\)
   D5.3 = 6. \(EEpEpEpEpEE qrEqrEEstE est\)
   D6.3 = 7. \(EEpqEpq\)
   D1.7 = 8. \(EEpEqrEsEsEsE EEq pr\)
   D2.8 = 9. \(EsEsEsEsEEpqEEprE prE pr\).

References


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