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RELATIONAL TERNARY SEMANTICS FOR A LOGIC EQUIVALENT TO INVOLUTIVE MONOIDAL t-NORM BASED LOGIC IMTL

Abstract

We define the logic ICI complete with respect to certain ternary relational structures. ICI is a particular negation extension of Urquhart's many-valued logic C. The logic ICI is deductively equivalent to Esteva and Godo's Involutive Monoidal t-norm based logic IMTL.

1. Introduction

In this paper, we define the logic ICI. This logic is a particular negation completion of Urquhart's well-known many-valued logic C (see [9], [10]). ICI is clearly sharply modelled in the ternary relational semantics, but, as it turns out, ICI is deductively equivalent to the logic IMTL formalized in [3] by Esteva and Godó. The logic ITML is an extension of the logic MTL defined in the same paper. MTL abbreviates Monoidal t-norm based logic and it is meant to be a basic system in the range of t-norm based logics. The logic IMTL (involutive Monoidal t-norm based logic) is the extension of MTL with the involutive negation axiom (INV)

$\neg\neg A \to A$

On the other hand, ICI is an extension of the logic CI formalized in [6]. The logic CI is the extension of Urquhart's positive many-valued logic

C (see[9], [10] and [4] for a correction on the logics C and CI) with the characteristic negation axioms of intuitionistic logic

$(A \to \neg B) \to (B \to \neg A)$
$\neg A \to (A \to B)$

Now, ICI (Involutive CI) is formalized from CI as IMTL was from MTL. That is, ICI is the result of extending CI with the involutive negation axiom INV.

In the concluding remarks of this paper, we shall comment on the relationship between IMTL (MTL) and ICI (CI) on the one hand, and with Ono's contractionless logics, on the other. We now turn to its structure. In §2-4, we prove the completeness of CI with respect of certain ternary relational structures. Although completeness of CI is proved in [6], the completeness result we present here is new. In [6], negation was intuitionistically modelled, i.e., it was understood as a special kind of conditional

$\neg A =_{df} A \to F$

where F is a falsity constant which is satisfied in none of the points of the model. In this paper, instead, we shall formalize negation by means of the "Routley star" $\star$. (This will ease the formalization of the negation in ICI, that is the first aim of this paper).

In connection with the relational ternary semantics presented here, let us make a couple of remarks. First, as it is well known, these semantics were in principle designed for relevance logics and theories in their environment need not be non-null, consistent or complete. Nevertheless, we will show how to adapt the semantics to the highly paradoxical logic ICI. In particular, we show how to treat the non-null, (absolutely) consistent theories required in the ICI models. Secondly, the operator $\star$ is always used in the relational ternary semantics to formalize involutive types of negation. We will show, however, how to use $\star$ to formalize the non-involutive negation that is characteristic of CI (and MTL).

In §5, 6 we formalize the logic ICI and in §7 we prove the deductive equivalence between IMTL and ICI. We shall prove this by using a result of Pei in [8] on the relationship between Wang's logic L_0^* and IMTL. Paragraph 8 is an appendix on the logics included in TW_+ (therefore in ICI) and the contraction and reductio axioms that, as it is known, are not valid in Łukasiewicz's logics in which ITML is included. (TW_+ is positive Ticket

Entailment without the contraction axiom (see, e.g., [1]). Finally, we state some brief concluding remarks.

2. The logic CI

The logic CI is axiomatized with

A1. $(A \to B) \to [(B \to C) \to (A \to C)]$
A2. $A \to [(A \to B) \to B]$
A3. $A \to (B \to A)$
A4. $(A \to B) \vee (B \to A)$
A5a. $(A \wedge B) \to A$
A5b. $(A \wedge B) \to B$
A6. $[(A \to B) \wedge (A \to C)] \to [A \to (B \wedge C)]$
A7a. $A \to (A \vee B)$
A7b. $B \to (A \vee B)$
A8. $[(A \to C) \wedge (B \to C)] \to [(A \vee B) \to C]$
A9. $[A \wedge (B \vee C)] \to [(A \wedge B) \vee (A \wedge C)]$
A10. $(A \to \neg B) \to (B \to \neg A)$

The rules of inference are *Modus ponens* (MP): if $\vdash A$ and $\vdash A \to B$, then $\vdash B$, and *Adjunction* (adj.): if $\vdash A$ and $\vdash B$, then $\vdash A \wedge B$.

Some representative theses of SW are:

T1.	$[A \to (B \to C)] \to [B \to (A \to C)]$	A1, A2
T2.	$B \to (A \to A)$	A3, T1
T3.	$A \to A$	T2
T4.	$(B \to C) \to [(A \to B) \to (A \to C)]$	A1, T1
T5.	$[(A \to B) \to C] \to (B \to C)$	A1, A3
T6.	$[(A \to B) \to (A \to C)] \to [B \to (A \to C)]$	T5
T7.	$[(A \to B) \to (A \to C)] \to [A \to (B \to C)]$	T1, T6
T8.	$[(A \to A) \to B] \to B$	A2, T3
T9.	$A \to [B \to (A \wedge B)]$	A3, A6, T2
T10.	$[(A \wedge B) \to C] \to [A \to (B \to C)]$	A1, T1, T9
T11.	$(A \to B) \to [(A \to C) \to [A \to (B \wedge C)]]$	A6, T10
T12.	$(A \to C) \to [(B \to C) \to [(A \vee B) \to C]]$	A8, T10
T13.	$(A \to B) \to [(A \wedge C) \to (B \wedge C)]$	A5, T1, T11
T14.	$(A \to B) \to [(A \vee C) \to (B \vee C)]$	A7, T1, T12

T15. $[A \to (B \wedge C)] \to [(A \to B) \wedge (A \to C)]$ A5, A6
T16. $[(A \vee B) \to C] \to [(A \to C) \wedge (B \to C)]$ A6, A7
T17. $[(A \to B) \vee (A \to C)] \to [A \to (B \vee C)]$ A7, A8
T18. $[(A \to C) \vee (B \to C)] \to [(A \wedge B) \to C]$ A5, A8
T19. $[A \to (B \vee C)] \to [(A \to C) \vee (B \to C)]$
A4, A7, A8, T1, T3, T12
T20. $[(A \wedge B) \to C] \to [(A \to C) \vee (B \to C)]$
A4, A7, A8, T1, T3, T11
T21. $(A \vee B) \leftrightarrow (B \vee A)$ A7, A8
T22. $(A \wedge B) \leftrightarrow (B \wedge A)$ A5, A6
T23. $[A \vee (B \vee C)] \leftrightarrow [(A \vee B) \vee C]$ A7, A8
T24. $[A \wedge (B \wedge C)] \leftrightarrow [(A \wedge B) \wedge C]$ A5, A6
T25. $A \leftrightarrow (A \vee A)$ A7, A8, T3
T26. $A \leftrightarrow (A \wedge A)$ A5, A6, T3
T27. $A \leftrightarrow [A \vee (A \wedge B)]$ A5, A7, A8, T3
T28. $A \leftrightarrow [A \wedge (A \vee B)]$ A5, A6, A7, T3
T29. $[A \wedge (B \vee C)] \leftrightarrow [(A \wedge B) \vee (A \wedge C)]$
A5, A6, A7, A8, A9
T30. $[A \vee (B \wedge C)] \leftrightarrow [(A \vee B) \wedge (A \vee C)]$
A5, A6, A7, A8, A9
T31. $A \to \neg\neg A$ A10
T32. $(A \to B) \to (\neg B \to \neg A)$ A10, T31
T33. $\neg A \to (A \to B)$ A3, A10
T34. $A \to (\neg A \to B)$ T1, T33
T35. $[B \to \neg (A \to A)] \leftrightarrow \neg B$ A10, T8, T33
T36. $\neg (A \vee B) \leftrightarrow (\neg A \wedge \neg B)$ A5, A6, A7, A8, A10, T32
T37. $(\neg A \vee \neg B) \to \neg (A \wedge B)$ A5, A8, T32
T38. $(A \vee B) \to \neg (\neg A \wedge \neg B)$ A10, T36
T39. $(A \wedge B) \to \neg (\neg A \vee \neg B)$ A10, T37
T40. $\neg (A \wedge B) \to (\neg A \vee \neg B)$ T20, T35
T41. $(A \vee B) \to (\neg A \to B)$ A3, T34
T42. $(\neg A \vee B) \to (A \to B)$ A3, T33
T43. $(\neg A \to B) \vee (A \to \neg B)$ A4, A10

3. Semantics for CI

A CI model is a quadruple $< K, R, \star, \models >$ where K is a non-empty set, R is a ternary relation on K and $\star$ an operation on K satisfying the following definitions and conditions.

For every a, b, c, $d \in K$ (quantifiers ranging over K):

d1. $a \leq b =_{df} \exists x Rxab$
d2. $R^2abcd =_{df} \exists x\,(Rabx\ \&\ Rxcd)$
P1. $a \leq a$
P2. $a \leq b\ \&\ Rbcd \Rightarrow Racd$
P3. $R^2abcd \Rightarrow \exists x\,(Racx\ \&\ Rbxd)$
P4. $Rabc \Rightarrow Rbac$
P5. $Rabc\ \&\ Rade \Rightarrow b \leq e$ or $d \leq c$
P6. $Rabc \Rightarrow Rac^\star b^\star$
P7. $a \leq a^{\star\star}$

Finally, $\models$ is a valuation relation from K to the sentences of the propositional language satisfying the following conditions for all formulas p, A, B and point a in K:

(i). $a \models p$ and $a \leq b \Rightarrow b \models p$
(ii). $a \models A \vee B$ iff $a \models A$ or $a \models B$
(iii). $a \models A \wedge B$ iff $a \models A$ and $a \models B$
(iv). $a \models A \rightarrow B$ iff for all $b, c \in K, (Rabc\ \&\ b \models A) \Rightarrow c \models B$
(v). $a \models \neg A$ iff $a^\star \not\models A$

A is valid in CI ($\models A$) iff $a \models A$ for all $a \in K$ in all models.

Some features of these semantics clearly distinguish them from the standard relevance semantics. One of them is, of course, postulate P5. But the really essential ones are the following three. Firstly, the definition of validity in respect of all the points in K , not in respect of some designated subset of K . Secondly, the definiton of the binary relation $\leq$ in respect of any element in K and not in respect of some designated subset of K. And finally, the fact that negation is not involutive. The first two features automatically verify the usual set of implicative paradoxes. On the other hand, we remark that the following conditions are fulfilled in all models:

P8. $(a \leq b \ \& \ b \leq c) \Rightarrow a \leq c$	P2, P4, d1
P8'. $d \leq b \ \& \ Rabc \Rightarrow Radc$	P2, P4
P9. $Rabc \Rightarrow b \leq c$	d1
P10. $Rabc \Rightarrow a \leq c$	P4, d1
P11. $R^2abcd \Rightarrow \exists x\,(Rbcx \ \& \ Raxd)$	P3, P4, d2
P12. $R^2abcd \Rightarrow R^2acbd$	P3, P4, d2
P13. $a \leq b \Rightarrow b^\star \leq a^\star$	P6, d1
P14. $Rabc^\star \Rightarrow Racb^\star$	P8', P6, P7
P15. $\exists x \ Ra^\star ax$	P1, P4, P6

We sketch a proof of the semantic consistency theorem (correctness). We have:

PROPOSITION 1. $(a \leq b \ \& \ a \models A) \Rightarrow b \models A$

PROOF. By induction on the length of A. Use P2 in the case of the conditional and P13 in the case of negative formulas. □

PROPOSITION 2. $\models A \rightarrow B$ *iff for all* $a \in K$ *in all models, if* $a \models A$*, then* $a \models B$.

PROOF. By P1 and Proposition 1. □

PROPOSITION 3. *If* $a \models A$*, then* $a^\star \not\models \neg A$

PROOF. By Clause (v) and P7. □

Then, we have:

THEOREM 1.(**semantic consistency**) *If* $\models A$*, then* $\models A$.

PROOF. Use Proposition 2 and the semantic postulates as follows: A6-A10, MP and adj. are trivial. Then, A1, A2, A3 and A4 are verified by P3, P4, P10 and P5, respectively. Finally, A10 is verified by P6 with the assistance of Proposition 3. □

We finish this section with the following note:

NOTE. We remark that mere clauses (ii), (iii) and (v) are sufficient for T41 $\neg(A \wedge B) \rightarrow (\neg A \vee \neg B)$ to be satisfied. But T41 is not, of course, intuitionistically valid. Given that the above mentioned clauses seem to be the adequate ones for the respective connectives, this obviously means that types of negation equal to or weaker than the intuitionistic one are not representable with the "Routley star" $\star$Thus, for example, we cannot provide a similar semantics to these developed in this paper for the intuitionistic logics of [7].

4. Completeness of CI

The canonical model is the structure $< K^C, R^C, \star^C, \models^C >$ where K^C is the set of all non-null prime consistent theories, R^C is such that for any a, $b \in K^C$, $R^C abc$ iff for all wff A, B, $(A \to B \in a \;\&\; A \in b) \Rightarrow B \in c$, $\star^C$ is such that for any $a \in K^C$, $a^{\star C}$ is defined like this: $a^{\star C} =_{df} \{A\colon \neg A \notin a\}$, and $\models^C$ is such that for any $a \in K^C$ and wff A, $a \models^C A$ iff $A \in a$.

A theory a is a set of wff closed by adjunction and provable entailment, i.e., a is a *theory* if (i) if $A, B \in a$, then $A \wedge B \in a$ and (ii) if $\vdash A \to B$ and $A \in a$, then $B \in a$. A theory is *regular* if it contains all the theorems of CI. A theory is *inconsistent* if it contains the negation of a theorem. Finally, a theory a is *non-null* if no wff belongs to it, and a theory a is *prime* if whenever $A \vee B \in a$, then $A \in a$ or $B \in a$.

Now, as it was pointed out in the Introduction, we have to adapt the standard argument for relevance logics (see, e.g., [1]) to the case of non-null, consistent theories. We prove:

Proposition 4. *If a is a non-null theory, then a is regular*

Proof. By A3 $\square$

Proposition 5. *If $A \in a$, then $\neg A \notin a^{\star C}$.*

Proof. By T31 and definitions. $\square$

Proposition 6. *If a is a prime theory, then $a^{\star C}$ is a prime theory.*

Proof. (i) $a^{\star C}$ is closed by provable entailment: by T32. (ii) $a^{\star C}$ is closed by adjunction: by T40. (iii) $a^{\star C}$ is prime: by T37. $\square$

Proposition 7. *A theory is inconsistent iff every wff belongs to it.*

Proof. From left to right (proof from right to left is obvious): suppose that a is an inconsistent theory, i.e., suppose $\neg B \in a$ for some theorem B. By T34, $\vdash \neg B \to C$. Consequently, $C \in a$ where C stands for any wff. $\square$

It is worth noting that contradiction does not necessarily entails inconsistency (we do not have $(A \wedge \neg A) \rightarrow B$) though the converse obviously holds.

PROPOSITION 8. *If a is a consistent theory, then $a^{\star C}$ is a non-null theory.*

PROOF. Let C be a theorem and $C \notin a^{\star C}$. Then, $\neg C \in a$ by definition. But a is consistent. $\square$

PROPOSITION 9. *If a is a regular theory, then $a^{\star C}$ is consistent.*

PROOF. Suppose $\neg B \in a^{\star C}$ for some theorem B. By Proposition 5, $B \notin a$. But a is regular. $\square$

PROPOSITION 10. *$\star^C$ is an operation on K^C,that is, if a is a non-null prime consistent theory, then so is $a^{\star C}$.*

PROOF. By Propositions 6, 8 and 9 $\square$

PROPOSITION 11. *If a and b are non-null theories, then the set $x = \{B\colon \exists A(A \rightarrow B \in a \text{ and } A \in b)\}$ is a non-null theory such that $R^T abx$ (R^T is the generalization of R^C to the set of all theories).*

PROOF. It is easy to prove that x is closed by adjunction and provable entailment. $R^T abx$ is obvious and that x is non-null is proved as follows. Suppose $\vdash B$ and $C \in b$. By A3, $\vdash C \rightarrow B$. As a is non-null, it is regular (Proposition 4), so $C \rightarrow B \in a$. By $R^T abx$, $B \in x$. $\square$

Note that if a and b are consistent theories and $R^T abx$, we do not generally have that x is a consistent theory. Actually, what we have is

PROPOSITION 12. *If $R^T abc$ and c is consistent, then a and b are consistent.*

PROOF. (i) Suppose b inconsistent. Then $\neg A \in b$ for some theorem A. By T34, $\vdash \neg A \rightarrow B$. As a is regular, $\neg A \rightarrow B \in a$. So, $B \in c$. But C represents any wff. (ii) Suppose a inconsistent. By $Rabc$ and P4, $Rbac$.Then, by a similar argument, C would be inconsistent. $\square$

Now, the proofs of the two following Propositions would be similar to the ones provided in relevance logics (see, e.g., [1]) Proposition 13. *For each non-theorem A there is some $x \in K^C$ such that $A \notin x$.*

Proposition 14. *For any a, $b \in K^C$, $a \leq^C b$ iff $a \subseteq b$*

Next, we have

Proposition 15. *The canonical valuation relation is a valuation relation.*

Proof. Clauses (i)-(iii) are trivial and clause (v) is immediate by definitions, so the clause of interest is (iv). From left to right, the proof is immediate. From right to left, the proof is as follows. Suppose $a \not\models A \rightarrow B$ for some $a \in K^C$ and some wff A, B. We prove that there are b, $c \in K^C$ such that $b \models A$ and $c \not\models B$. We define the sets $x = \{B\colon \vdash A \rightarrow B\}$ and $y = \{B\colon C \rightarrow B \in a$ and $c \in x\}$ that are non-null theories such that $R^T axy$, $x \models A$, $y \not\models B$. Then, we prove that x and y are consistent.

(i) *x is consistent*: Suppose $\neg C$ for some theorem C. By definition of x, $\vdash A \rightarrow \neg C$. By A10, $\vdash C \rightarrow \neg A$. As $C \in a$ (a is regular), $\neg A \in a$. By T33, contradicting our hypothesis, $A \rightarrow B \in a$.

(ii) *y is consistent*: Suppose $\neg C$ for some theorem C. By definition of y, $D \rightarrow \neg C \in a$ for some $D \in x$.By definition of x, $\vdash A \rightarrow D$. By A1, $\vdash (D \rightarrow \neg C) \rightarrow (A \rightarrow \neg C)$. So, $A \rightarrow \neg C \in a$. By A2 and the theoremhood of C, $\vdash (C \rightarrow \neg A) \rightarrow \neg A$. Then, $\neg A \in a$ by A10. Whence, by T33, $A \rightarrow B \in a$, which contradicts our hypothesis.

Finally, x and y are extended to prime theories b and c in the customary way. □

Proposition 16. *The semantical conditions P1-P6 hold in the canonical model.*

Proof. P1, P2, P3, P4, P6 are proved just as in the standard semantics and P5 easily follows with A4. Next, P7 is proved by Proposition 5 and definitions and finally, P3 is proved as follows. Suppose $R^{C2}abcd$. Define $y = \{B\colon A \rightarrow B \in a$ and $a \in c\}$. The set y is a non-null theory such that $Racy$ (Proposition 11). Moreover, $Rbyd$ holds by A1. Now, the proof that y is consistent is similar to the proof given for x in Proposition 15 (i). Next, y is extended to a prime theory e such that $Race$ and $Rbed$. □

By Propositions 13, 15 and 16, we have

Proposition 17. *The canonical model is in fact a model.*

Finally, by Propositions 13 and 17 we have

Theorem 2. (**Completeness of CI**) *If* $\models A$, $\vdash A$

5. The logic ICI

The logic ICI (Involutive CI) is the result of adding the axiom

A11. $\neg\neg A \to A$

to CI. In addition to T1-T44, the following are theorems of ICI:

T45. $(\neg A \to \neg B) \to (B \to A)$	A10, A11
T46. $(\neg A \to B) \to (\neg B \to A)$	T31, T45
T47. $\neg(\neg A \wedge \neg B) \to (A \vee B)$	T37, T46
T48. $\neg(\neg A \vee \neg B) \to (A \wedge B)$	T41, T46
T49. $\neg(A \to B) \to (A \wedge \neg B)$	A11, T37, T43
T50. $\neg(A \to \neg B) \to (A \wedge B)$	A11, T49

Also, we have:

Proposition 18. *Any of T45-T50 can axiomatize ICI instead of A11.*

Proof. A11 is derivable from each one of these theorems as follows. T46: by T3; T45: deduce T46 with T31; T47, T48: $\neg\neg A \to \neg(\neg A \wedge \neg A)$, $\neg\neg A \to \neg(\neg A \vee \neg A)$ are immediate. T49, T50: by T35. □

6. Semantics for ICI

A ICI model is the same as a CI model save for the addition of the condition:

P16. $a^{\star\star} \leq a$

We remark that the following conditions are fulfilled in all models:

P17. $Rab^{\star}c \Rightarrow Rac^{\star}b$	P6, P15
P18. $Ra^{\star}bc \Rightarrow Rbc^{\star}a$	P4, P17
P19. $Ra^{\star}bc \Rightarrow b \leq a$	P10, P18
P20. $Ra^{\star}bc \Rightarrow c^{\star} \leq a$	P9, P17

In order to prove the semantic consistency of ICI, we note that Proposition 3 can be strengthened to

PROPOSITION 19. *$a^\star \models \neg A$ iff $a \not\models A$.*

PROOF. By clause (v), P7 and P16. □

Then we have:

THEOREM 3. (**Semantic consistency of ICI**) *If $\vdash A$, then $\models A$.*

PROOF. We just have to prove that A11 is valid: use Proposition 19 and P16. □

To prove the completeness of ICI we note that Proposition 5 can be strengthned to

PROPOSITION 20. *$\neg A \in a^{\star C}$ iff $A \notin a$.*

PROOF. By T31, A11 and definitions. □

Finally, we have:

THEOREM 4. (**Completeness of ICI**) *If $\models A$, then $\vdash A$.*

PROOF. We just have to prove that P16 holds in the canonical model: use Proposition 20. □

7. Deductive equivalence between ICI and IMTL

In this section we prove that ICI and IMTL are deductively equivalent logics. Wang introduces and motivates in [11] the fuzzy propositional calculus L_0^*. This logic is axiomatized as follows (see [8]; we refer to the list of axioms and theorems in §2 and §5): A3, A7, A8, T1, T2, T9, T11, T20, T21, T31, T45 and modus ponens as the sole rule of inference. Now, Pei proves that L_0^* and IMTL are equivalent logics ([8], Theorem 7). Therefore we have:

PROPOSITION 21. *L_0^* and IMTL are deductively included in ICI.*

On the other hand, A1, A3, A4 and A5-A11 of ICI are theorems of IMTL ([3], Proposition 1), and A2, A11 and adjunction of ICI are trivially proved in IMTL as follows (we refer to [3]):

A2. $A \to [(A \to B) \to B]$: from A7b, $[(\varphi \& \psi) \to \chi] \to [\varphi \to (\psi \to \chi)]$ and $[\varphi \& (\psi \to \chi)] \to \psi$ (Proposition 1, (4)).

A11. $(\neg A \to \neg B) \to (B \to A)$: from $(\neg\varphi \to \neg\psi) \to (\psi \to \varphi)$ (Proposition 4, (36)) and $\neg\neg\varphi \to \psi$ (INV, p.276).

Adjunction (if $\vdash A$ and $\vdash B$, then $\vdash A \wedge B$): from $\varphi \to [\psi \to (\varphi \& \psi)]$ and $(\varphi \& \psi) \to (\varphi \wedge \psi)$ (Proposition 1, (5), (9)). Therefore we have:

Proposition 22. *ICI is deductively included in IMTL.*

And from propositions 19 and 20:

Proposition 23. *L_0^*, IMTL and ICI are deductively equivalent.*

Let us examine these logics from another point of view. The logic MTL referred to in the introduction to this paper can be axiomatized dropping the involutive double negation axiom INV from IMTL. Now, MTL is equivalent to Ono's well-known contractionless logic Flew extended with the linearity axiom (Lin) $(\varphi \to \psi) \vee (\psi \to \varphi)$, Flew [Lin] ([5], pp. 18-19). Then, the result of adding INV to Flew is equivalent to ICI (L_0^* , IMTL).

We finish this section with the following note:

Note. It can directly be proved that IMTL is deductively included in ICI. We define

$A \,\&\, B =_{df} \neg(A \to \neg B)$
$\bar{O} =_{df} \neg(A \to A)$

where & and $\bar{O}$ are the strong conjunction and the truth constant of IMTL, respectively, and $\to$, $\neg$ the conditional and the negation of ICI. Then, it is not difficult to prove the axioms of ITML in which & or $\bar{O}$ appear, to wit ([3], p.273):

A2. $(\varphi \& \psi) \to \varphi$
A3. $(\varphi \& \psi) \to (\psi \& \varphi)$
A6. $[\varphi \& (\varphi \to \psi)] \to (\varphi \wedge \psi)$
A7a. $[\varphi \to (\psi \to \chi)] \to [(\varphi \& \psi) \to \chi]$
A7b. $[(\varphi \& \psi) \to \psi] \to [\varphi \to (\psi \to \chi)]$
A9. $\bar{O} \to \varphi$

Finally the axiom

A8. $[(\varphi \to \psi) \to \chi] \to [[(\varphi \to \psi) \to \chi] \to \chi]$

is immediate from A4, T1 and A12 of ICI.

8. Concluding remarks

Let us look at CI from alternative perspectives. Ciabattoni defines in [2] the logic C$^\sharp$ and proves by proof-theoretical methods that C$^\sharp$ is decidable. Well, it would be easy to prove that C$^\sharp$ and CI are equivalent logics. Still, from another point of view, CI is Ono's contractionless logic Flew without fusion $\circ$ and extended with the linearity axiom A4 of §2, i.e., CI is equivalent to (the fusion-free fragment of) Flew [Lin] (see [4], pp. 18-19). Then, given the equivalence between MTL and Flew [Lin] (see §6), CI would, in principle, be equivalent to the strong conjunction-free fragment of MTL. Now, we think that the results in this paper help to clarify the relationship between relevance and many-valued logics and, so, between relational semantics and algebraic semantics. In this sense, it would certainly be interesting to formalize in the ternary relational semantics the extension of CI equivalent to MTL (or Flew [Lin]) as well as the semantics for other extensions of MTL (or Flew [Lin]) or IMTL (or Flew plus the axiom INV) (see [3] or [4] for a survey of these extensions).

A. An appendix on the contraction and reductio axioms

Consider the following contraction and reductio axioms:

t1. $[A \to (A \to B)] \to (A \to B)$
t2. $[A \to (B \to C)] \to [(A \to B) \to (A \to C)]$
t3. $(A \to B) \to [[A \to (B \to C)] \to (A \to C)]$
t4. $[A \to (B \to C)] \to [(A \wedge B) \to C]$
t5. $\neg (A \wedge \neg A)$
t6. $A \vee \neg A$
t7. $[A \to (B \wedge \neg B)] \to \neg A$
t8. If $\vdash A \to B$ and $\vdash A \to \neg B$, then $\neg A$

t9. $(A \wedge \neg A) \to B$
t10. $(A \to \neg A) \to \neg A$
t11. $(\neg A \to A) \to A$
t12. $(A \to B) \to [(A \to \neg B) \to \neg A]$
t13. $(\neg A \to B) \to [(\neg A \to \neg B) \to \neg A]$
t14. $(A \to B) \to [(\neg A \to B) \to B]$
t15. $(A \wedge B) \to \neg (A \to \neg B)$
t16. $(\neg A \to B) \to (A \vee B)$
t17. $[(A \to B) \to A] \to A$

These axioms t1-t17 are not valid in Łukasiewicz's three valued logic. Therefore, they are not theorems of ICI.

Now, let TW_+ (Positive Ticket Entailment without contraction) be axiomatized with A1, T2, T3, A5-A9 and MP and adj. as rules of inference. TW (Ticket Entailment without contraction) is the result of adding A10 and A11 to TW_+; EW_+ (Positive Entailment logic without contraction) is the result of adding the assertion rule (asser.) $\vdash A \Rightarrow \vdash (A \to B) \to B$ to TW_+, and finally, EW (Entailment logic without contraction) is the result of adding asser. to TW. (cfr., e.g., [1]).

We remark that all the logics we have just defined are included in ICI. We have (proofs are left to the reader):

Proposition 24. *Given TW, t10-t16 are equivalent*

Proposition 25. *Given TW_+, t1 is derivable from t3 or t4*

Proposition 26. *Given TW_+, t1 is derivable from t17*

Proposition 27. *Given TW, t5-t9 are equivalent*

Proposition 28. *Let us add the thesis $(A \to B) \to (A \to A)$ to TW, then t5 and t10 are equivalent*

It is immediate from these propositions that if any of the axioms (t1-t16) is added to ICI, either the contraction axiom t1 or else the reductio axiom t8 is derivable. But it is clear that we have

Proposition 29. *ICI plus t1 or t8 is classical propositional logic.*

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