A NOTE ON GLOBALLY ADMISSIBLE INFERENCE RULES FOR MODAL AND SUPERINTUITIONISTIC LOGICS

Abstract

In this shot note we consider globally admissible inference rules. A rule $r$ is globally admissible in a logic $\mathcal{L}$ if $r$ is admissible in all logics with the finite model property which extend $\mathcal{L}$. Here we prove a reduction theorem: we show that, for any modal logic $\mathcal{L}$ extending $K4$, a rule $r$ is globally admissible in $\mathcal{L}$ iff $r$ is admissible in all tabular logics extending $\mathcal{L}$. The similar result holds for superintuitionistic logics.

1. Introduction

The concept of admissible inference rules was introduced by Lorenzen (1955, cf. [4]). The investigation of such rules has various directions. For modal and superintuitionistic logics, this research (in terms of universal algebra) is a study of quasi-equational theories of free modal and pseudo-boolean algebras.

An important problem is to find an algorithm determining admissible inference rules. This problem was initially addressed to the intuitionistic propositional calculus IPC (cf. H.Friedman [3]). First affirmative solution of this question was found in [5]. Descriptions of bases for admissible rules is another interesting direction. The question about existence of finite bases for rules admissible in IPC was set by A.Kuznetsov.

It turns out that, for many important (we would say basic) non-standard logics, there are no bases for admissible rules in finitely many

In this note we would like to extend the study of admissible inference rules and to explore inference rules which, for a given logic $L$, are admissible in all logics extending $L$ and possessing the fmp. We call such rules globally admissible in $L$. The main result of this note is a theorem describing rules globally admissible in any logic $L$ (where $L$ is a modal logic extending $K4$ or a superintuitionistic logic) by reduction the question to admissibility of rules in all tabular logics over $L$.

2. Definitions, Notation, Preliminaries

We assume the basic facts concerning Kripke semantics for modal and superintuitionistic logics, inference rules and their admissibility are known to the reader (though we recall briefly below all necessary facts). As a reference we would recommend Chagrov and Zakharyaschev [1] for general technique, and Rybakov [10] for advanced technique concerning inference rules. To recall definitions, a frame $F := (F, R)$ is a pair, where $F$ is a nonempty set and $R$ is a binary relation on $F$. In this note, we will often denote the base set of a frame $F$ and the frame itself by the same letter. In what follows the relation $R$ in frames is always a transitive relation or a partial order. Any subset $C$ of a frame $F$, which consists of all mutually $R$-comparable elements of $F$ or which is an irreflexive element, is a cluster of $F$. For any element $a \in F$ the cluster containing $a$ is denoted by $C(a)$.

A model based on a frame $(F, R)$ is a triple $(F, R, V)$, where $V$ is a valuation of a set of propositional letters. Truth values of formulas $\beta$ at elements $a$ of $F$ w.r.t. $V$ is defined inductively (as usual). The maximal number of clusters in $R$-chains of clusters arising from an element $x$ (a cluster $C$) is the depth of the element $x$ (the cluster $C$).

For any transitive frame $F$ (or transitive Kripke model $M$) its $n$-slice, $- Sl_n(F)$ ($Sl_n(M)$) – is the set of all elements of depth $n$ from $F$ ($M$). $S_n(F)$ is the set of all elements from $F$ with depth at most $n$. 
For $b \in F$, $b^R := \{x|\exists y \in C(b) : yRx\} \cup C(b)$; $b^<_R := \{x|\exists y \in C(b) : yRx \land \neg(xRy)\}$. For any $X \subseteq F$, $X^R := \cup \{x^R|x \in X\}$. For any model $\mathcal{M} := (F, V)$ and any formula $\alpha$ with variables from the domain of $V$ and any $x \in F$, $x \models_V \alpha$ is the denotation for $\alpha$ is true at $x$ in the model $\mathcal{M}$ w.r.t. $V$. If we need to distinguish the model where we consider the truth relation then we write $(\mathcal{M}, x) \models_V \alpha$.

For frames (or models) $\mathcal{M}_1$ and $\mathcal{M}_2$, the denotation $\mathcal{M}_1 \sqsubseteq \mathcal{M}_2$ means $\mathcal{M}_1$ is an open subframe (or, respectively, an open submodel) of $\mathcal{M}_2$. If $\mathcal{M}_1$ is an open submodel of $\mathcal{M}_2$, and $x \in \mathcal{M}_1$ then, for any formula $\alpha$, $(\mathcal{M}_1, x) \models_V \alpha$ iff $(\mathcal{M}_2, x) \models_V \alpha$ (well known fact).

A frame $\mathcal{F} := (F, R)$ is rooted, if $\exists a \in F$ such that $\forall b \in F (aRb)$, then we say $a$ is the root of $\mathcal{F}$ and denote it by $\text{root}(\mathcal{F})$. A (structural) inference rule $\forall$, is a sequent $\alpha_1, \ldots, \alpha_n \beta$, where $\alpha_1, \ldots, \alpha_n, \beta$ are some formulas.

A rule $r = \alpha_1(x_1, \ldots, x_n), \ldots, \alpha_k(x_1, \ldots, x_n)/\beta(x_1, \ldots, x_n)$ is admissible in a logic $\mathcal{L}$ if the following holds. For arbitrary formulas $\delta_1, \ldots, \delta_n$, if $\forall j(a_j(\delta_1, \ldots, \delta_n)) \in L$ hold true, then $\beta(\delta_1, \ldots, \delta_n) \in L$.

A Kripke model $\langle W, R, V \rangle$, where $V : \{p_1, p_2, \ldots, p_n\} \rightarrow 2^W$, is $n$-characterizing for a logic $\mathcal{L}$ if, for any formula $\alpha$ in variables $p_1, \ldots, p_n$, $\alpha \in L$ iff $\langle W, R, V \rangle \models \alpha$.

A description of special $n$-characterizing models $\text{Ch}_L(n)$ for logics $\mathcal{L}$ with the finite model property and criteria for recognizing admissibility of inference rules in $L$ based on these models can be found, for instance, in [10]. We will use these techniques, therefore we recall briefly the construction of $\text{Ch}_L(n)$ and the semantic criterion for recognizing admissibility.

For any element $a$ of a Kripke model $\mathcal{M}$ with a valuation $V$, $V(a)$ is the set of all propositional letters which are true w.r.t $V$ at $a$. Let $\mathcal{L}$ be any logic extending $K4$ which has the finite model property. Given a set $P_n := \{p_1, \ldots, p_n\}$ of propositional letters, we construct the first slice $S_1(\text{Ch}_L(n))$ of the model $\text{Ch}_L(n)$ as follows. $S_1(\text{Ch}_L(n))$ consists of the collection of all clusters $C$ which are $\mathcal{L}$-frames with all possible valuations $V$ of letters from $P_n$, where clusters do not have doubling - elements with the same valuation, i.e. $\forall x, y \in C(x \neq y \Rightarrow V(x) \neq V(y))$. 
Assuming $S_m(Ch_L(n))$ – the set of all elements of depth $m$ from $Ch_L(n)$ – to be constructed, we extend $S_m(Ch_L(n))$ by adjoining new set $Sl_{m+1}(Ch_L(n))$. We put in $Sl_{m+1}(Ch_L(n))$ the elements as follows.

Take all antichains $Y$ of elements from $S_m(Ch_L(n))$, where any $Y$ has at least one element of depth $m$. Next, for any $Y$, we put in $Sl_{m+1}(Ch_L(n))$ all clusters $C$ from $S_1(Ch_L(n))$, assuming $C$ to be $R$-immediate predecessor for all elements from $Y$, where (i) any $C$ generate as the root $L$-frame $C_R$, (ii) if $Y$ consists of a single cluster which is not an irreflexive element, then $C$ is not a submodel of $Y$. Iterating this procedure, we get as the result the model $Ch_L(n)$. We need the following facts.

**Theorem 1.** (cf. for instance, [10], Theorem 3.3.11) For any logic $L \supseteq K4$, the model $Ch_L(n)$ is $n$-characterizing for $L$.

For a frame $F$, a valuation $V$ in $F$ and an inference rule $r := \alpha_1, \ldots, \alpha_n/\beta$, we say $r$ is valid in $F$ w.r.t. $V$, and write $F \models_V r$ if the following holds. If $\forall x \in F$ and $\forall \alpha_i (x \models_V \alpha_i)$ holds, then $\forall x \in F (x \models_V \beta)$. A rule $r$ is valid in a frame $F$ if $r$ is valid in $F$ w.r.t. any valuation, we write then $F \models r$.

For a model $M := \langle M, R, V \rangle$, a valuation $V_1$ of a certain set of letters $x$ is said to be definable in $M$ if, for any $x$ from the domain $Dom(V_1)$ of $V_1$, there is a formula $\alpha_x$ constructed from letters of $Dom(V)$ such that $V_1(x) = V(\alpha_x)$.

**Theorem 2.** (Theorem 3.3.3 [10]) For any inference rule $r$, $r$ is admissible in $L$ iff, for any $n$, $r$ is valid in the frame of $Ch_L(n)$ w.r.t. any definable valuation.

Similar results hold for superintuitionistic logics. The construction of $n$-characterizing models is the lightened version of given one above for modal logics. In this case we consider only partial orders $\leq$ instead of transitive relations $R$ and only intuitionistic valuations instead of arbitrary ones.

3. A Description of Globally Admissible Inference Rules

We say a rule $r$ is globally admissible in a logic $L$ if $r$ is admissible in all logics which extend $L$ and have fmp.
Theorem 3. Let $\mathcal{L}$ be a modal logic extending K4. A rule $r$ is globally admissible in $\mathcal{L}$ iff $r$ is admissible in all tabular logics extending $\mathcal{L}$.

Proof. If a rule $r$ is globally admissible in $\mathcal{L}$ then obviously $r$ is admissible in any tabular logic extending $\mathcal{L}$. To prove the opposite, assume a rule $r := \alpha_1, ..., \alpha_m/\beta$ is admissible in all tabular logics extending $\mathcal{L}$, but is not admissible in a logic $\mathcal{L}_1 \supseteq \mathcal{L}$, where $\mathcal{L}_1$ has fmp.

Then by Theorem 2, for some $n$, the rule $r$ is refuted in the frame of the $n$-characterizing model $Ch_{\mathcal{L}_1}(n)$ for $\mathcal{L}_1$ by a definable valuation $V$. Thus,

(i) $Ch_{\mathcal{L}_1}(n) \models_{V} (\bigwedge_{1 \leq i \leq m} \alpha_i)$ but $Ch_{\mathcal{L}_1}(n) \not\models_{V} \beta$. So,

(ii) $(Ch_{\mathcal{L}_1}(n), x) \not\models_{V} \beta$

for some $x \in Ch_{\mathcal{L}_1}(n)$.

By construction of $Ch_{\mathcal{L}_1}(n)$ the frame $x^R$ is a finite rooted frame and $\mathcal{L}(x^R) \supseteq \mathcal{L}_1 \supseteq \mathcal{L}$. Consider the tabular logic $\mathcal{L}_2 := \mathcal{L}(x^R)$. By construction of $Ch_{\mathcal{L}_1}(n)$ the frame of the model $Ch_{\mathcal{L}_1}(n)$ is an open subframe of the frame of $Ch_{\mathcal{L}_1}(n)$. The valuation $V$ is definable in $Ch_{\mathcal{L}_1}(n)$, therefore the restriction $V_2$ of $V$ to $Ch_{\mathcal{L}_2}(n) \subseteq Ch_{\mathcal{L}_1}(n)$ is also definable in $Ch_{\mathcal{L}_2}(n)$ (truth values for formulas in models and their open submodels are the same). In accordance with the construction of $Ch_{\mathcal{L}_2}(n)$ the cone $(x^R)$ from $Ch_{\mathcal{L}_1}(n)$ occurs in $Ch_{\mathcal{L}_2}(n)$ as an open submodel. Thus using (i) and (ii) we conclude

(iii) $Ch_{\mathcal{L}_2}(n) \models_{V_1} (\bigwedge_{1 \leq i \leq m} \alpha_i)$ but $Ch_{\mathcal{L}_2}(n) \not\models_{V_1} \beta$.

The valuation $V_2$, as we pointed above, is definable in $Ch_{\mathcal{L}_2}(n)$. Therefore by Theorem 2 the rule $r$ is not admissible in $\mathcal{L}_2$, $\mathcal{L}_2$ is tabular and extends $\mathcal{L}$, a contradiction. $\square$

Theorem 4. A rule $r$ is globally admissible in a superintuitionistic logic $\mathcal{L}$ iff $r$ is admissible in all tabular logics extending $\mathcal{L}$.

Proof. is a variant of the previous one adopted for superintuitionistic logics. $\square$
Thus, in order to describe all rules globally admissible in logics with fmp, we can restrict ourselves by consideration only tabular logics.

**Consequence 1.** If a rule $r$ is admissible in all tabular modal (superintuitionistic) logics extending a modal logic $\mathcal{L} \supseteq K4$ (a superintuitionistic logic $\mathcal{L}$), where $\mathcal{L}$ has fmp, then $r$ is admissible in $\mathcal{L}$.

So, in particular, the recognizing of admissibility of rules in a logic $\mathcal{L}$ with fmp can be reduced to determination of admissibility in all tabular logics extending $\mathcal{L}$.

Now we would like slightly refine these general results in order to restrict the class of tabular logics at which we have to verify the admissibility of rules.

**Theorem 5.** A rule $r$ is admissible in all tabular modal (superintuitionistic) logics extending a modal (superintuitionistic) logic $\mathcal{L}$ iff $r$ is admissible in all tabular logics generated by rooted finite frames $\mathcal{F}$, where $\mathcal{L}(\mathcal{F}) \supseteq \mathcal{L}$.

**Proof.** From the left to the right is evident. The proof of the converse can be easily extracted from the proof of Theorem 3. □

So, in sum, we reduce the question about global admissibility of inference rules in a logic $\mathcal{L}$ with fmp to admissibility of rules in all tabular logics extending $\mathcal{L}$ generated by finite rooted frames. An open interesting question is to find algorithms which recognize rules globally admissible in distinguished non-classical logics, such as IPC or $S4$. For this we have to describe rules which are admissible in all corresponding tabular logics.

**References**


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