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REMARKS ON HERTZ ALGEBRAS AND IMPLICATIVE SEMILATTICES

Abstract
It is a fact that several authors have accepted as evident that the class of Hilbert algebras with the property that for each pair of elements there exists its infimum are the same as H. Porta’s Hertz algebras, H. Curry’s implicative semilattices or P. Köhler’s Brouwerian semilattices. In [2], J. Cirulis pointed out that in some cases this statement has led to incorrect conclusions.
After having analyzed such irregularities, we aim to point out that such confusions and inaccuracies are minor and remediable ones and they do not invalidate the importance of the interesting papers they appear in. Besides, we show how such "mistakes" may be simply eliminated.

Hilbert algebras were introduced by L. Henkin ([9]) in dual form, as an algebraic counterpart of Hilbert’s positive implicative propositional calculus ([20], ch. II), and they were intensively studied by A. Diego in [6] (see also [5]).

Following A. Diego ([6]), by a Hilbert algebra, we mean an algebra \( \langle A, \rightarrow \rangle \) of type 2 satisfying the following identities:

\[
\begin{align*}
D_1: & \quad x \rightarrow x = y \rightarrow y, \\
D_2: & \quad (x \rightarrow x) \rightarrow x = x, \\
D_3: & \quad x \rightarrow (y \rightarrow z) = (x \rightarrow y) \rightarrow (x \rightarrow z), \\
D_4: & \quad (x \rightarrow y) \rightarrow ((y \rightarrow x) \rightarrow x) = (y \rightarrow x) \rightarrow ((x \rightarrow y) \rightarrow y).
\end{align*}
\]

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Denoting by 1 the fix element \( x \rightarrow x \), every Hilbert algebra is made into a poset \((A, \leq)\) with greatest element 1, under the ordering \( x \leq y \) if and only if \( x \rightarrow y = 1 \).

In 1963, H. Porta in [18], Definition 1.4, introduced the notion of Hertz algebras as we will reproduce below:

*If a Hilbert algebra \((A, \rightarrow)\) is a meet-semilattice relative to the natural ordering \(\leq\), we will say that \(A\) is a Hertz algebra.*

Next to this definition, Porta asserted that Hertz algebras can also be defined as systems \(\langle A, \rightarrow, \wedge \rangle\), where \(A\) is a set, and \(\rightarrow, \wedge\) are two binary operations on \(A\) satisfying the following identities:

\[
\begin{align*}
M_1: \quad & x \rightarrow x = y \rightarrow y, \\
M_2: \quad & (x \rightarrow y) \wedge y = y, \\
M_3: \quad & x \rightarrow (y \wedge z) = (x \rightarrow z) \wedge (x \rightarrow y), \\
M_4: \quad & x \wedge (x \rightarrow y) = x \wedge y.
\end{align*}
\]

But in fact, it is well–known that there are Hertz algebras in the sense of Definition 1.4 ([18]), which according to [7] we shall call Hilbert algebras with infimum, where some of the identities \(M_1\) to \(M_4\) are not valid. To this end, we lead the reader to the bibliography quoted in [2], [7] and [12]. The easiest example to show this last assertion is the following one:

\[
\begin{array}{c|cccc}
\wedge & 0 & a & b & 1 \\
\hline
0 & 0 & 0 & 0 & 1 \\
a & 0 & 1 & b & 1 \\
b & 0 & a & 1 & 1 \\
1 & 0 & a & b & 1 \\
\end{array}
\]

\[
\begin{array}{c|cccc}
\rightarrow & 0 & a & b & 1 \\
\hline
0 & 1 & 1 & 1 & 1 \\
a & 0 & 1 & b & 1 \\
b & 0 & a & 1 & 1 \\
1 & 0 & a & b & 1 \\
\end{array}
\]

\[
\begin{array}{c|cccc}
0 & 0 & 0 & 0 & 0 \\
a & 0 & a & 0 & 1 \\
b & 0 & 0 & b & b \\
1 & 0 & a & b & 1 \\
\end{array}
\]

It is worth noting that the poset of the above example has been considered in 1961 by A. Diego ([5], [6]) in order to show that in Hilbert algebras the partial order relation doesn’t determine the implication operation \(\rightarrow\).
On the other hand, it seems to be right to suppose that Porta also realized the mistake above mentioned because later in 1981, in his paper [19], containing among others the proofs of the results announced in [18], he omitted Definition 1.4 and called Hertz algebras the algebras $\langle A, \rightarrow, \wedge \rangle$ of type $(2, 2)$ which verify $M_1$ to $M_4$.

In 1955, A. Monteiro proved in Theorem 3 ([14], page 159) that the implicative systems which have been introduced by H. Curry in [3], under the name of implicative logical groups, are the same as the algebras $\langle A, \rightarrow, \wedge \rangle$ of type $(2, 2)$ which satisfy the identities $M_1$ to $M_4$. A few years later, in 1958, A. Monteiro ([15]) started naming them Hertz algebras instead of implicative systems.

In short, we intended to put in evidence that the name of Hertz algebras is due to A. Monteiro and that they coincide with the implicative semilattices considered by H. Curry ([4]). To the best of our knowledge, it was in [18] the first time this name appeared in the literature. Besides, since H. Porta is a distinguished mathematician, it seems natural that several authors have considered without a doubt Definition 1.4 as equivalent to the equational one, in agreement to his assertion established in [18].

On the other hand, in [7] the authors studied the class of Hilbert algebras $\langle A, \rightarrow, 1 \rangle$ with the property that for every $x, y \in A$ there exists the infimum $x \wedge y$ of $\{x, y\}$ concerning to the natural ordering $\leq$. Adding the binary operation $\wedge$ to the operations on $A$, they introduced a new equational class of algebras $\langle A, \rightarrow, \wedge, 1 \rangle$ which they named $iH-$algebras. More precisely, $iH-$algebras are defined both by means of the above pointed identities $D_1$ to $D_4$ and the following ones:

\begin{align*}
iH_1: \quad & x \wedge (y \wedge z) = (x \wedge y) \wedge z, \\
iH_2: \quad & x \wedge x = x, \\
iH_3: \quad & x \wedge (x \rightarrow y) = x \wedge y, \\
iH_4: \quad & (x \rightarrow (y \wedge z)) \rightarrow ((x \rightarrow z) \wedge (x \rightarrow y)) = 1.
\end{align*}

Besides, it is worth remarking that from [7], Lemma 2.2, it can be proved that the class of $iH-$algebras which satisfy the additional identity:

\begin{align*}
iH_5: \quad & x \rightarrow (y \rightarrow (x \wedge y)) = 1,
\end{align*}

coincide with the variety of implicative semilattices (see [8], [12]).
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J. Cirulis in [2], just like H. Porta in [18], D. Busneag in [1] and M. Kondo in [11], considered the notion of Hertz algebras as in [18], Definition 1.4, and taking this statement into account, he appropriately asserted that not only [1] but also [11] have mistakes. To this fact we can say:

(i) In [10] the notion of (H)-Hilbert algebras was introduced as Hilbert algebras which satisfy the following condition:

\[ H: \text{ the set } A(a,b) = \{ x \in A : a \to (b \to x) = 1 \} \text{ has least element} \]
for all \( a, b \in A \) and this one is denoted by \( a + b \).

Afterwards, M. Kondo in [1], Theorem 1, proved that the class of implicative semilattices is the same as that of (H)-Hilbert algebras where \( a + b \) is the infimum of \( \{a, b\} \). But as it was well established in [2], taking into account Definition 1.4 in [18], this result is incorrect. Just as the referee of this paper has indicated, the proof of Theorem 1 is a direct consequence of the fact that the class of Hertz algebras coincides with that of implicative semilattices, and that the same holds for the class of (H)-Hilbert algebras according to what was established in [2] and [12]. However, our purpose has been to follow the M. Kondo’s reasoning in obtaining Theorem 1, and so, after analysing his proof, we indicate that everything is solved by completing the proof of Lemma 2. For this end, it only remains to verify that identity iH5 holds true, which is simple to do.

(ii) In [1], D. Busneag, based on the results established in [18], asserted the equivalence between Definition 1.4 and the equational one given below this. In Lemma 3.6, the same author considered Definition 1.4, and since the latter is not valid, it seems to be possible to doubt whether his result is true or false according to the correct meaning of the term Hertz algebras. However, as the operations \( \to \) and \( \wedge \) for multipliers on \( H \) (see [1], Section 3) are defined pointwise, then it is simple to verify iH5 since this axiom holds true in \( H \).

Besides, D. Busneag included [16], among others, as a reference work on Hertz algebras but in this paper there is no mention of them.

Finally, in order to render a simple homage to Professor Antonio Monteiro, our long range aim is to emphasize that in the 60’s, he was one of the authors who made a fundamental contribution to the development of
the theory of Hilbert algebras and implicative semilattices. Some of these results were obtained many years ago, independently by other well-known mathematicians not familiar with Monteiro’s work.

J. Cirulis asserted that Marsden ([12], [13]) obtained a lot of results on Hilbert algebras, specially those about filters, also known as deductive systems. But as a matter of fact, Monteiro was the one who introduced this last notion and determined almost all the results known about them and their relationship with congruences. Besides A. Diego, one of his disciple, was the one who applied this profusely in his important paper [6].

References


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