COMBINING TIME AND KNOWLEDGE,
SEMANTIC APPROACH

Abstract
The paper investigates a semantic approach for combining knowledge and time. We introduce a multi-modal logic $L(T_K)$ containing modalities for knowledge and time in a semantic way, as the set of all $T_K$-valid formulae for a class of special frames $T_K$. The main result of our paper is the theorem stating that $L(T_K)$ is decidable and giving a resolving algorithm. The result is proven by using standard tools: filtration, bulldozing and contracting $p$-morphisms.

1. Introduction
The paper is devoted to study a semantic approach to model knowledge and time. Study of time and knowledge within framework of modal logic is an active area nowadays (cf. [2], [3], [4], [6] and references therein). Sound and complete axiomatizations for a number of different logics involving modalities for knowledge and time are found in [4]. Our approach, in a sense, is from an opposite site: we generate a logic combining knowledge and time in a semantic manner, via a class of frames which defines such a logic. Our aim is to study the question about decidability. We would like to investigate to which extend a standard technique of modal logic works, we like to construct a deciding algorithm using only standard technique of modal logics without involving heavy technique as automatons or the Rabin theorem.
We model the time as a linear discrete sequence of time states, and the knowledge is represented by a tuple of modal-like operations $K_i$ (imitating knowledge of agents) which operates in time states containing a set of information nodes. We start by introduction of a certain class of multi-modal Kripke frames which have the structure described above and generate the logic $L(TK)$ as the set of all formulae which are valid in these frames. We assume time flow to be linear and discrete and agents operating synchronously: they have access to a sort of shared clock\(^3\), each agent knowing what time it is and distinguishing present from future time. The main result of our paper is the theorem stating that $L(TK)$ is decidable and giving a resolving algorithm.

2. Notation, Definitions

General notation and definitions concerning modal logics which we will use can be found, for instance, in [1], [5]. To study the combination of knowledge and time we will use the language of multi-modal logic. Our language $L^{TK}$ is chosen as follows: the alphabet of $L^{TK}$ contains propositional letters $P := \{p_1, ..., p_n, ...\}$, round brackets $(,)$, standard boolean operations, and the set of modal operations $\{\Box, \Diamond, \{K_i \mid i \in I := \{1, ..., k\}\}$. Well formed formulae (wff) are defined in the standard way, in particular, if $A$ is a wff, then $\Box A$, $\Diamond A$, $K_i A$, for all $i \in I$, are wff. $\text{Fma}(L)$ is the set of all well formed formulae of $L^{TK}$. The informal meaning of the modal operations is as follows. The set $I := \{1, ..., k\}$ indicates $k$ distinct agents. $\Box A$ means: the formula $A$ will always be true; $K_i A$: the agent $i$ knows $A$ in the current time state and the current information node; $\Diamond A$: the wise agent knows $A$ in the current time state and current information node.

Semantics for this language is based on linear and discrete time flow, associating a time point with any natural number $n$. As semantic tools we will use the following Kripke-Hintikka frames: $T_K := (W_{TK}, R_{\leq}, R_{\prec}, R_1, ..., R_k)$, where the base set of $T_K$ is the disjoint union of sets $\mathcal{C}^n$, $W_{TK} := \bigcup_{n \in \mathbb{N}} \mathcal{C}^n$. Binary relations $R_{\leq}$, $R_{\prec}$, and $R_1, ..., R_k$ are as follows: $R_{\leq}$ is the following linear, reflexive and transitive relation on $W_{TK} \times W_{TK}$:

\[
\forall x, y \in W_{TK} (x R_{\leq} y \text{ iff } \exists n_1, n_2 \in \mathbb{N} ((x \in \mathcal{C}^{n_1}) \& \& (y \in \mathcal{C}^{n_2}) \& (n_1 \leq n_2)));
\]

\(^3\)See Fagin et al., [2], pp. 127-128.
$R_n$ is the equivalence relation on any $C^n \in W_{\mathcal{K}}$:
\[ \forall x, y \in W_{\mathcal{K}}, (xR_n y) \iff \exists n \in \mathbb{N} (x \in C^n \& y \in C^n); \]
Any $R_n$ is some equivalence relation on any $C^n$.

The informal meaning of these frames is as follows. Any cluster $C^n$ contains a set of information nodes available at the time point $n$. The relation $R_n$ is the connection of the information nodes by time current: $xR_n y$ indicates that the node $y$ is a node available in the same time as $x$, or $y$ is an information node in a future time point. $xR_n y$ says that $x$ and $y$ are nodes in the same time point, and $xR_n y$ indicates that in the current time point $y$ is accessible from $x$ by of the agent $i$ authorities. A model $\mathcal{M}_{\mathcal{K}}$ on $\mathcal{T}_{\mathcal{K}}$ is a tuple $\mathcal{M}_{\mathcal{K}} = \langle \mathcal{T}_{\mathcal{K}}, V \rangle$ where $V$ is a valuation of a set $P$ of propositional letters in $\mathcal{T}_{\mathcal{K}}$. That is, for any $p_i \in P$ $V(p_i) \subseteq W_{\mathcal{K}}$.

The valuation $V$ can be extended from the set $P$ onto all wff’s constructed from $P$ in the standard way. In particular, $\forall x \in W_{\mathcal{K}},$
\[ x \models_{V} \Box_x A \iff \forall y \in W_{\mathcal{K}} (xR_n y \Rightarrow y \models_{V} A); \]
\[ x \models_{V} \Diamond_x A \iff \forall y \in W_{\mathcal{K}} (xR_n y \Rightarrow y \models_{V} A); \]
\[ x \models_{V} K_i A \iff \forall y \in W_{\mathcal{K}} (xR_n y \Rightarrow y \models_{V} A). \]

Let $\mathcal{M}_{\mathcal{K}} := \langle \mathcal{T}_{\mathcal{K}}, V \rangle$ be a model on a frame $\mathcal{T}_{\mathcal{K}}$; a formula $A \in Fma(\mathcal{L}_{\mathcal{K}})$ is said to be true in $\mathcal{M}_{\mathcal{K}}$ at the point $a \in W_{\mathcal{K}}$ if $a \models_{V} A$. A formula $A$ is true in the model $\mathcal{M}_{\mathcal{K}}$, notation $\mathcal{M}_{\mathcal{K}} \models A$, if $\forall a \in W_{\mathcal{K}}, a \models_{V} A$. $A$ is valid in the frame $\mathcal{T}_{\mathcal{K}}$ notation $\mathcal{T}_{\mathcal{K}} \models A$, if, for any model $\mathcal{M}_{\mathcal{K}}$ on $\mathcal{T}_{\mathcal{K}}$, $\mathcal{M}_{\mathcal{K}} \models A$.

**Definition 2.1.** The logic $L(\mathcal{T}_{\mathcal{K}})$ is the set of all $\mathcal{T}_{\mathcal{K}}$-valid formulae:
\[ L(\mathcal{T}_{\mathcal{K}}) := \{ A \in Fma(\mathcal{L}_{\mathcal{K}}) \mid \mathcal{T}_{\mathcal{K}} \models A, \forall \mathcal{T}_{\mathcal{K}}\text{-frame} \} \]

### 3. Decidability

The aim of our paper is to prove that the logic $L(\mathcal{T}_{\mathcal{K}})$ is decidable. Initially we will show that any formula $A$ which is not a theorem of $L(\mathcal{T}_{\mathcal{K}})$ can be refuted by a frame similar to $\mathcal{T}_{\mathcal{K}}$ but of a finite size computable from the length of $A$. Consider and fix for the rest of this paper a formula $A$ such that $A \notin L(\mathcal{T}_{\mathcal{K}})$. Then there is a frame $\mathcal{T}_{\mathcal{K}}$ and a model $\mathcal{M}_{\mathcal{K}} := \langle \mathcal{T}_{\mathcal{K}}, V \rangle$ based on this frame such that, $\exists a \in W_{\mathcal{K}}, (\mathcal{M}_{\mathcal{K}}, a) \not\models_{V} A$. Firstly we reduce the number of elements in any $C^n$ to a finite number of ones effectively bounded from size of $A$. This can be easy done by a standard filtration
on any separate $C^n$. Below we briefly sketch this technique. Let $\text{Sub}(A)$ be the set of all the sub-formulae of $A$. Define the equivalence relation $\approx$ on $W_{\mathcal{T}_n}$ as follows: $\forall a, b \in W_{\mathcal{T}_n} [a \approx b$ iff $\exists n \in \mathbb{N} (a, b \in C^n$ & $\forall \beta \in \text{Sub}(A) (a \models_{\mathcal{V}} \beta$ iff $b \models_{\mathcal{V}} \beta))].$ Next, define the quotient set of the original model: $\forall a \in W_{\mathcal{T}_n} [a]_{\approx} := \{b \mid a \approx b\}, \forall n \in \mathbb{N} C^n_{\approx} := \{[a]_{\approx} \mid a \in C^n\}, W_{\mathcal{T}_n}^{\approx} := \bigcup_{n \in \mathbb{N}} C^n_{\approx}$.

The model resulting from this filtration is based on this quotient set and looks as follows: $\mathcal{M}_{\mathcal{T}_n}^{\approx} := \langle W_{\mathcal{T}_n}^{\approx}, R^n_{\approx}, R^n_{\approx}, ..., R^n_{\approx}, V^{\approx}\rangle$ where: $\forall p \in \text{Sub}(A)$, $V^{\approx}(p) := \{[a]_{\approx} \mid a \in V(p)\}; \forall [a]_{\approx}, [b]_{\approx} \in W_{\mathcal{T}_n}^{\approx}$,

\[ [a]_{\approx} R^n_{\approx} [b]_{\approx} \text{ iff } \exists n, m \in \mathbb{N} ([a]_{\approx} \in C^n_{\approx} \land [b]_{\approx} \in C^n_{\approx} \land n \leq m); \]

\[ [a]_{\approx} R^n_{\approx} [b]_{\approx} \text{ iff } \exists n, m \in \mathbb{N} ([a]_{\approx} \in C^n_{\approx} \land [b]_{\approx} \in C^n_{\approx} \land n = m); \]

$\forall i \in I [a]_{\approx} R^n_i [b]_{\approx}$ iff $\exists n \in \mathbb{N} ([a]_{\approx}, [b]_{\approx} \in C^n_{\approx}$ & $\forall \beta \in \text{Sub}(A) ((\mathcal{M}_{\mathcal{T}_n}, a) \models_{\mathcal{V}} K_i \beta$ iff $(\mathcal{M}_{\mathcal{T}_n}, b) \models_{\mathcal{V}} K_i \beta).$ Since the model described is a result of filtration the standard filtration-lemma holds:

**Lemma 3.1.** For any formula $\beta \in \text{Sub}(A)$, for any element $a \in W_{\mathcal{T}_n}$ $(\mathcal{M}_{\mathcal{T}_n}, a) \models_{\mathcal{V}} \beta$ iff $(\mathcal{M}_{\mathcal{T}_n}^{\approx}, [a]_{\approx}) \models_{\mathcal{V}^{\approx}} \beta.$

**Corollary 3.2.** $\mathcal{M}_{\mathcal{T}_n}^{\approx} \models_{\mathcal{V}^{\approx}} A.$

**Lemma 3.3.** If $\|\text{Sub}(A)\| := m$, then $\forall n \in \mathbb{N}, \|C^n_{\approx}\|$ is at most $2^m$.

Thus the model $\mathcal{M}_{\mathcal{T}_n}^{\approx}$ refutes $A$ and has clusters $C^n$ of effectively bounded size. Using $\mathcal{M}_{\mathcal{T}_n}^{\approx}$ we will construct a finite model refuting $A$. The clusters $C^0_{\approx}$ and $C^1_{\approx}$ are isomorphic (we will use in the sequel notation: $C^0_{\approx} \cong C^1_{\approx}$) if and only if there is a function $f$ s.t.: $f : C^0_{\approx} \rightarrow C^1_{\approx}$, (1) $f$ is a bijection, (2) $\forall \xi \in \{\preceq, \sim, 1, ..., k\}, \forall a, b \in C^0_{\approx} (a R^n\xi b$ iff $f(a) R^n\xi f(b))$, (3) $\forall p \in \text{Sub}(A), \forall a \in C^0_{\approx} (a \in V^{\approx}(p)$ iff $f(a) \in V^{\approx}(p))$. By Lemma 3.3 we conclude

**Proposition 3.4.** There is only a finite, computable from $A$, number of non-isomorphic time-clusters $C^n_{\approx} \in W_{\mathcal{T}_n}^{\approx}$.

For any time cluster $C^n_{\preceq}$, $C^m_{\preceq}$ is the set of all the $\preceq$-successor clusters of $C^n_{\preceq} : \forall C^n_{\preceq} \in W_{\mathcal{T}_n}, C^n_{\preceq} := \{C^j_{\preceq} \mid n \leq j\}, \text{ and } C^n_{\preceq}^+ := \bigcup C^n_{\preceq}$. In the sequel, $C^n_{\preceq}^+(M)$ or $C^n_{\preceq}^-(M)$ are described sets from a frame $M$ (we will alter these frames $M$).
DEFINITION 3.5. The time-cluster \( C^n_\approx \) is a stabilizing cluster if and only if for any \( C^n_\approx \), where \( n \leq j \), the sets \( C^n_\approx \) and \( C^j_\approx \) coincide up to isomorphism of clusters.

LEMMA 3.6. The model \( M^2_{\approx k} \) has a stabilizing cluster \( C^a \).

PROOF. By Proposition 3.4 the number of non-isomorphic time-clusters \( C^n_\approx \in W^\approx \approx_k \) is finite. The following also holds: \( \forall n, j \in \mathbb{N}, n \leq j \Rightarrow C^n_\approx \supseteq C^j_\approx \). Consider the sequence of all the time-clusters \( C^1_\approx, C^2_\approx, \ldots \). We construct a subsequence \( C^m_\approx \) of the sequence \( C^n_\approx, n \in \mathbb{N} \) as follows. Take \( C^1_\approx \); if \( C^1_\approx \) is a stabilizing cluster, then we stop, and the subsequence is chosen. Assume a subsequence \( C^1_\approx, \ldots, C^m_\approx \) is chosen. If \( C^m_\approx \) is not a stabilizing cluster, then there is a cluster \( C^k_\approx \), where, up to isomorphism, \( C^n_\approx \supseteq C^k_\approx \).

Take the \( \approx \)-smallest \( C^m_\approx \) with this property and set \( C^m_\approx := C^k_\approx \). Since \( C^m_\approx \supseteq C^{n+1}_\approx \), this procedure must terminate, and it terminates at a stabilizing cluster. \( \Box \)

We denote by \( C^a \) the \( \approx \)-smallest stabilizing cluster.

LEMMA 3.7.

If \( C^a \) is a stabilizing cluster, then \( \forall n, j \in \mathbb{N}, n, j \geq s \), the following holds. If \( C^n_\approx \) is isomorphic to \( C^j_\approx \) by a mapping \( f \), then \( \forall \beta \in \text{Sub}(A), \forall a \in C^n_\approx (\langle C^n_\approx, a \rangle \models_{V^\beta} \beta \iff \langle C^j_\approx, f(a) \rangle \models_{V^\beta} \beta) \).

Proof may be given by induction on the length of \( \beta \). The only non-trivial steps are the ones for the modal operations. If \( \beta \) is \( \Box \_B \) or \( K_i \_B \) for \( i \in I \) the claim holds by the induction hypotheses and the definition of isomorphism. Let \( \beta \) be \( \Box \_B \). Assume \( (C^n_\approx, a) \models_{V^\beta} \Box \_B \). We can have 3 cases: (i) \( n = j \) where the proof is trivial, (ii) \( n < j \), and (iii) \( n > j \). If \( n < j \), \( (C^n_\approx, a) \models_{V^\beta} \Box \_B \) implies that for any \( b \in C^n_\approx \) \( (b \models_{V^\beta} B) \). Since \( n < j \), \( C^n_\approx \supseteq C^j_\approx \) holds and \( \forall c \in C^j_\approx, (\langle M^2_{\approx k}, c \rangle \models_{V^\beta} B) \). Consequently \( (M^2_{\approx k}, f(a)) \models_{V^\beta} \Box \_B \) and \( (C^j_\approx, f(a)) \models_{V^\beta} \Box \_B \). The proof of the converse is similar to the case (iii) below. Consider the case (iii) when \( n > j \). Assume \( (C^n_\approx, a) \models_{V^\beta} \Box \_B \). This implies that, for any \( b \in C^n_\approx \), \( (\langle M^2_{\approx k}, b \rangle \models_{V^\beta} B) \). Since \( n, j \geq s \) and \( C^a \) is the stabilizing cluster, for any \( C^n_\approx \in C^n_\approx \) there is some \( C^m_\approx \in C^n_\approx \) such that \( C^m_\approx \cong C^m_\approx \). Therefore by induction hypothesis we conclude \( \forall C^m_\approx \in \)}
Therefore by IH we have the set of all $C^1 \subseteq C^n$, $(C^m, c) \models \mathcal{V}_= B$. Then $(\mathcal{M}_{T_k}^{\mathcal{R}}, f(a)) \models \mathcal{V}_= \Box \_B$ and $(C^1, f(a)) \models \mathcal{V}_= \square \_B$. The proof of the converse is similar to the previous case. \qed

For any time-cluster $C^n$, where $n \geq s$, $[C^n]_{\mathcal{R}}$ is the set of all the time-clusters isomorphic to $C^n$: $\forall n, j \geq s [C^n]_{\mathcal{R}} := \{ C^n | C^n \cong C^j \}$. Take and fix, for any $[C^n]_{\mathcal{R}}$ a unique representative cluster $C_n$. Let $St := \bigcup_{n \geq s} C_n$ be the set of all the elements of such clusters. We define a new finite model as follows: $\mathcal{M}_{T_k}^{\mathcal{R}} := (W_{T_k}, R_{T_k}^{\mathcal{R}}, \ldots, R_{T_k}^{\mathcal{R}}, C_{\mathcal{R}})$, where $W_{T_k}^{\mathcal{R}} := \{ C^1, C^2, \ldots, C^n, St \}$, $\forall p \in \text{Sub}(A)$ $V^p := \{ a \in W_{T_k}^{\mathcal{R}} | a \in V^p(p) \}$, $\forall a, b \in W_{T_k}^{\mathcal{R}} \forall n, j \leq s ((a \in C^n & b \in C^j) \implies (aR^2 b \text{ iff } aR^n b))$, otherwise, if $n, j > s$, $R^s_{T_k}$ is a universal relation on $St$: $\forall a, b \in St (aR^s b)$. And $aR^s b$ iff $aR^n b$, $\forall i \in I aR^s b$.

Lemma 3.8. For any formula $\beta \in \text{Sub}(A)$ and, for any $a \in W_{T_k}^{\mathcal{R}}$, $(\mathcal{M}_{T_k}^{\mathcal{R}}, a) \models \mathcal{V}_= \beta$ iff $(\mathcal{M}_{T_k}^{\mathcal{R}}, a) \models \mathcal{V}_= \beta$.

Proof is given by induction on the length of $\beta$. The steps for the boolean operations are standard. Let $\beta$ be $\Box \_B$. Assume $(\mathcal{M}_{T_k}^{\mathcal{R}}, a) \models \mathcal{V}_= \Box \_B$. Since $a \in C^n$ for some $n \in \mathbb{N}$, we have 2 cases: (A): $n \leq s$ and (B): $n > s$.

Consider (A). Then $(\mathcal{M}_{T_k}^{\mathcal{R}}, a) \models \mathcal{V}_= \Box \_B$ implies that for all elements $b \in C^n$ $(\mathcal{M}_{T_k}^{\mathcal{R}}, b) \models \mathcal{V}_= B$. $C^n$ belongs to $\mathcal{M}_{T_k}^{\mathcal{R}}$ by assumption. Let $C^n \subseteq \mathcal{M}_{T_k}^{\mathcal{R}}$ be the set of all $\preceq$-successors of $C^n$ in $\mathcal{M}_{T_k}^{\mathcal{R}}$ and $C^n \subseteq (\mathcal{M}_{T_k}^{\mathcal{R}})$ be the set of all $\preceq$-successors of $C^n$ in $\mathcal{M}_{T_k}^{\mathcal{R}}$. Then $C^n \subseteq (\mathcal{M}_{T_k}^{\mathcal{R}})$ implies $C^n \subseteq (\mathcal{M}_{T_k}^{\mathcal{R}})$.

Therefore by IH we have $\forall c \in C^n \subseteq (\mathcal{M}_{T_k}^{\mathcal{R}})$, $(\mathcal{M}_{T_k}^{\mathcal{R}}, c) \models \mathcal{V}_= B$, and so it follows $(\mathcal{M}_{T_k}^{\mathcal{R}}, a) \models \mathcal{V}_= \Box \_B$.

Consider the case (B): $n \geq s$. Then $(\mathcal{M}_{T_k}^{\mathcal{R}}, a) \models \mathcal{V}_= \Box \_B$ implies that $\forall b \in C^n$ $(\mathcal{M}_{T_k}^{\mathcal{R}}, b) \models \mathcal{V}_= B$. Consider all the clusters between $C^n$ and $C^n$: by the definition of stabilizing cluster, each of them is isomorphic to some cluster belonging to $C^n$. Therefore, by Lemma 3.7 we have that $\forall c \in C^n (s \leq j \leq n \implies (\mathcal{M}_{T_k}^{\mathcal{R}}, c) \models \mathcal{V}_= B)$. So we have $\forall b \in C^n \subseteq (\mathcal{M}_{T_k}^{\mathcal{R}}, b) \models \mathcal{V}_= B$. Since $St \subseteq C^n$, $\forall c \in St (\mathcal{M}_{T_k}^{\mathcal{R}}, c) \models \mathcal{V}_= B$ holds. Applying IH we conclude $\forall c \in St (\mathcal{M}_{T_k}^{\mathcal{R}}, b) \models \mathcal{V}_= B$ and it follows $(\mathcal{M}_{T_k}^{\mathcal{R}}, a) \models \mathcal{V}_= \Box \_B$.

Assume now that $(\mathcal{M}_{T_k}^{\mathcal{R}}, a) \models \mathcal{V}_= \Box \_B$. Since $a \in C^n$ for some $n \in \mathbb{N}$, we still have 2 cases: (C): $n \leq s$ and (D): $n > s$. In the case (C), when
A statement is given, and the proof is divided into several cases. For each case, a new model is constructed by dropping some elements from the previous model. The induction hypotheses are applied to conclude the statements.

For any formula $\beta$, the analogous cluster for $\beta$ is immediate because the relations $\forall T$ and $K$ are the same in $M_{Tc}$ and $M_{Tc}^B$. By Lemma 3.7, we have $\forall C^m \in [C_2^m] \forall C^m \in C^m$, $(M_{Tc}, b) \models V = B$ and so we can conclude $\forall c \in C^m$, $(M_{Tc}, c) \models V = B$. Consequently $(M_{Tc}, a) \models V = B$.

Consider now the case (D): $n > s$. $(M_{Tc}, a) \models V = B$ implies that $\forall b \in St (M_{Tc}, b) \models V = B$, because $R_{Tc}$ is an equivalence relation on $St \times St$. The rest of the proof for this case is similar to the final part of the case (C).

The inductive step for the case when $\beta$ is $\Box \beta$ or $\beta$ is $K_i \beta$, $i \in I$ is immediate because the relations $R^m_{Tc}$ and $R^B_{Tc}$ are the same in $M_{Tc}^m$ and $M_{Tc}^B$.

Thus, by this lemma, the model, $M_{Tc}$, is finite and refutes the formula $A$. Since the number of elements in this model is not effectively bounded, we do not have yet decidability of the logic $L(Tc)$. Below we will construct a new model by dropping some $\preceq$-clusters from $M_{Tc}$.

For any sub-formula $\beta$ of $A$, $C_2$ is the $\preceq$-maximal $\preceq$-cluster among $C_2, C_2^m, \ldots, C_2$ s.t. $\exists b \in C_2(M_{Tc}, b) \models V = B$ if such cluster exists. $C_{\preceq}$ is the analogous cluster for $\preceq$. The new model is as follows:

$$W_{Tc} := \bigcup_{\beta \in \text{Sub}(A)} C_{\beta} \cup \bigcup_{\beta \in \text{Sub}(A)} C_{\preceq \beta} \cup \text{St},$$

$$M_{Tc} := (W_{Tc}, R^c_{Tc}, R^2_{Tc}, \ldots, R^n_{Tc}, V^c)$$

where: $\forall p \in \text{Sub}(A)$ $V^c(p) := \{a \in W_{Tc} \mid a \in V^c(p)\}$, $\forall a, b \in W_{Tc}$, $\forall R_\xi \in \{R_\xi, R_\xi, R_1, \ldots, R_\xi\}$, $a R_\xi b$ iff $a R^B_{\xi} b$.

**Lemma 3.9.** For any formula $\beta \in \text{Sub}(A)$, for any element $a \in W_{Tc}$, $(M_{Tc}, a) \models V = B$ if $a R_{\xi} b$.

$(M_{Tc}, a) \models V = B$. The new model is as follows:
Proof is by induction on the length of $\beta$. Evidently we only need to consider the steps for modal operations. If $\beta$ is $\Box_2 B$ or $\beta$ is $K_i B$, the steps are evident because all the relations $R^F_\sim$ and $R^F_i$ are the same in $M^F_{T_K}$ and $M^B_{T_K}$. Consider the case when $\beta$ is $\Box_2 B$ or $\beta$ is $K_i B$. If $(M^F_{T_K}, a) \forces_{VF} \Box_2 B$ then $\forall b \in W^F_{T_K} (a R_\sim b \implies (M^F_{T_K}, b) \forces_{VF} B)$. Since $W^F_{T_K} \subseteq W^B_{T_K}$, by induction hypothesis we have $\forall c \in W^F_{T_K} (a R_\sim b \implies (M^F_{T_K}, b) \forces_{VF} B)$ and so $(M^F_{T_K}, a) \forces_{VF} \Box_2 B$.

If $(M^F_{T_K}, a) \not\models_{VF} B$ then there is an element $b \in W^F_{T_K}$ such that $a R_\sim b$ and $(M^F_{T_K}, b) \not\models_{VF} B$.

If $b \in St$, then clearly $(M^F_{T_K}, b) \not\models_{VF} B$ and $(M^F_{T_K}, a) \not\models_{VF} B$. Otherwise there is an $R_\sim$-maximal cluster $C_{\sim}B$ among $C^1_\sim, C^2_\sim, \ldots, C^s_\sim$ and a $c \in C_{\sim}B$ such that $(M^F_{T_K}, c) \not\models_{VF} B$. Since $C_{\sim}B$ belongs to $W^F_{T_K}$ by IH we conclude $(M^F_{T_K}, c) \not\models_{VF} B$. Since $a R_\sim b$, it follows $(M^F_{T_K}, a) \not\models_{VF} \Box_2 B$.

So, by this lemma A is refuted by the model $M^F_{T_K}$ with effectively bounded size. Take an arbitrary frame $F$ with the structure as the frame of a model $M^F_{T_K}$. It is easy to show that $F$ is a $p$-morphic image of a frame $T_K$ based on $\sim$-clusters from the $\sim$-linear part of $F$ which are $\sim$-followed by an infinite chain of $\sim$-clusters subsequently doubling the remaining part of $\sim$-clusters from $F$. Therefore all theorems of $L(T_K)$ are true in $F$, and we have the following

**Theorem 3.10.** The logic $L(T_K)$ has the finite model property with computable size of refuting models, and hence $L(T_K)$ is decidable.

**References**


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