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THE LOGIC $K$ OF STRICT IMPLICATION
AND ITS RELATIVES

Abstract

The logics of strict implication (cf. [2]) are usually defined on the standard propositional language $\mathcal{L} = (\mathcal{L}, \land, \lor, \rightarrow, \neg)$ in which the sign "\rightarrow" is interpreted as a connective of strict implication. At times, in order to formulate only essential properties of pure strict implication and to avoid some possible influence or even distortion from the other connectives, a purely implicational language $\mathcal{L} = (\mathcal{L}, \rightarrow)$ seems to be more appropriate. In this paper I am going to consider three logics of strict implication defined on such a language. The main aim of the paper is to provide a set of axioms adequate for the logic of strict implication corresponding to modal logic $K$. Moreover, the paper contains some new sets of axioms for the logics $T$ and $S4$ of strict implication, obtained rather non-standard method – without application of Lindenbaum lemma.

1. The logic $K$ of strict implication.

We are going to consider a propositional language with the only connective being strict implication, i.e. $\mathcal{L} = (\mathcal{L}, \rightarrow)$. Let $\mathit{Var}$ be a set of the propositional variables of the language $\mathcal{L}$. Let us take a class $M$ of all Kripke models $m = < W, r, v, |=_m >$ (cf. [1]) such that
1. $W$ is any non-empty set (of points);
2. $r$ is any binary relation on $W : r \subseteq W \times W$;
3. $v : \mathit{Var} \rightarrow P(W)$, is any function;
4. $|=_m \subseteq W \times L$ is a binary relation defined as follows: for any $x \in W, \alpha, \beta \in L, p \in \mathit{Var}$
4a. $x \models_m p$ iff $x \in v(p)$.

4b. $x \models_m \alpha \to \beta$ iff $\forall y \in W[\forall x, y \geq r \Rightarrow (y \models_m \alpha \Rightarrow y \models_m \beta)]$.

As usual, the expression “$x \models_m \alpha$” means that the formula $\alpha$ is true at point $x$ in the model $m$.

The consequence relation $\models^M \subseteq P(L) \times L$ of the logic $K$ of strict implication is defined semantically in the standard way by the class of models $M$:

$$X \models^M \alpha \iff \forall m \in M \forall x \in W[\forall \beta \in X, x \models_m \beta \Rightarrow x \models_m \alpha].$$

In order to define an adequate set of rules of inference for $\models^M$ let us consider the smallest set $Ax$ of formulas closed on the following rules:

$(A) : \alpha \to \alpha$.

$(R) : \alpha / \beta \to \alpha$.

$(R_n) : \alpha_n \to (\alpha_{n-1} \to \ldots \to (\alpha_1 \to (\alpha \to \beta)))\ldots$,

$$\alpha_n \to (\alpha_{n-1} \to \ldots \to (\alpha_1 \to (\alpha \to \gamma)))\ldots, \ n = 0, 1, 2, 3,\ldots;$$

in particular, the rule $(R_0)$ is of the form: $\alpha \to \beta, \beta \to \gamma / \alpha \to \gamma$.

Let us consider the derivability relation $\vdash_R$ defined by the following set of rules: $R = \{(Ax), (R_0), (R_1), (R_2)\ldots\}$, where $(Ax)$ is an axiomatic rule of the form: $<\emptyset, \alpha > : \alpha \in Ax$.

Now we show that both logics, $\models^M$ and $\vdash_R$, coincide.

One can prove the soundness theorem: $\vdash_R \subseteq \models^M$.

In order to prove the completeness let us start from the following property of the relation $\vdash_R$:

**Lemma 1:** For any theory $X$ of the logic $\vdash_R$ and any formulas $\alpha, \beta$:

$$\alpha \to \beta \in X \iff \forall \gamma \in \text{Th}(\vdash_R)(<X, \gamma > \in \rho \& \alpha \in Y \Rightarrow \beta \in Y), \text{ where } \rho \text{ is a binary relation defined on the set } \text{Th}(\vdash_R) \text{ of all theories of the logic } \vdash_R \text{ as follows: } \forall X, \gamma \in \text{Th}(\vdash_R), <X, \gamma > \in \rho \iff \forall \alpha, \beta \in L (\alpha \to \beta \in X \& \alpha \in Y \Rightarrow \beta \in Y).$$

**Proof:** ($\Rightarrow$): by definition of the relation $\rho$.

($\Leftarrow$): Assume that $\alpha \to \beta \notin X$, where $X$ is a theory of the logic $\vdash_R$. Let us put $Y = \{\gamma : \alpha \to \gamma \in X\}$. First notice that $\beta \notin Y$ and $\alpha \in Y$ (since $\alpha \to \alpha \in Ax \subseteq X$). Now we show that $Y$ is closed on every rule from $R$, i.e., $Y \in \text{Th}(\vdash_R)$. 

Then from the definition of \( Ax \) let us consider the class \( X \). Moreover, \( X \in Th(\vdash_R) \), so \( X \) is closed on the axiomatic rule \( (Ax) \), thus \( Ax \subseteq X \), that is \( \alpha \rightarrow \gamma \in X \). Therefore \( \gamma \in Y \) by definition of \( Y \). In this way \( Y \) is closed on the rule \( (Ax) \). In order to show that \( Y \) is closed on \( (R_n) \), \( n = 0, 1, 2 \ldots \), assume that

\[
\alpha_n \rightarrow (\alpha_{n-1} \rightarrow \ldots \rightarrow (\alpha_1 \rightarrow (\alpha_0 \rightarrow \beta_0)) \ldots ) \in Y \quad \text{and} \quad \\
\alpha_n \rightarrow (\alpha_{n-1} \rightarrow \ldots \rightarrow (\alpha_1 \rightarrow (\beta_0 \rightarrow \gamma_0)) \ldots ) \in Y.
\]

Then from the definition of \( Y \) it follows that

\[
\alpha \rightarrow (\alpha_n \rightarrow (\alpha_{n-1} \rightarrow \ldots \rightarrow (\alpha_1 \rightarrow (\alpha_0 \rightarrow \beta_0)) \ldots )) \in X \quad \text{and} \quad \\
\alpha \rightarrow (\alpha_n \rightarrow (\alpha_{n-1} \rightarrow \ldots \rightarrow (\alpha_1 \rightarrow (\alpha_0 \rightarrow \gamma_0)) \ldots )) \in X.
\]

But \( X \) is closed on \( (R_{n+1}) \), so

\[
\alpha \rightarrow (\alpha_n \rightarrow (\alpha_{n-1} \rightarrow \ldots \rightarrow (\alpha_1 \rightarrow (\alpha_0 \rightarrow \beta_0)) \ldots )) \in X \quad \text{and} \quad \\
\alpha \rightarrow (\alpha_n \rightarrow (\alpha_{n-1} \rightarrow \ldots \rightarrow (\alpha_1 \rightarrow (\alpha_0 \rightarrow \gamma_0)) \ldots )) \in X.
\]

In order to complete the proof it is enough to show that \( <X, Y> \in \varrho \).

Suppose that \( \alpha \rightarrow \beta_0 \in X \) and \( \alpha_0 \in Y \). From the definition of the set \( Y \) we have \( \alpha \rightarrow \alpha_0 \in X \). Then \( \alpha \rightarrow \beta_0 \in X \), since \( X \) is closed on \( (R_0) \). Therefore \( \beta_0 \in Y \).

Now let us consider the following canonical model \( k \) belonging to the class \( M : k =<Th(\vdash_R), \varrho, v, \models_k> \), where \( \varrho \) is the binary relation defined in Lemma 1 and \( v : Var \rightarrow P(Th(\vdash_R)) \) is a function such that \( v(p) = \{X \in Th(\vdash_R) : p \in X \} \), for any \( p \in Var \). Naturally, relation \( \models_k \) is defined like in every model from the class \( M \). Then we have

**Lemma 2:** \( \forall X \in Th(\vdash_R)(X \models_k \alpha \iff \alpha \in X) \).

**Proof:** Standard, based on Lemma 1.

**The strong completeness theorem:** \( \models^M \subseteq \vdash_R \)

**Proof:** Assume that \( X \not\vdash_R \alpha \). Let us put \( Y = \{ \beta \in L : X \vdash_R \beta \} \in Th(\vdash_R) \). Therefore, from the assumption: \( \alpha \notin Y \). Obviously \( X \subseteq Y \).

Therefore, due to Lemma 2, in the canonical model, we have: \( Y \not\models_k \alpha \) and 

\( \forall \beta \in X : Y \models_k \beta \). So \( X \not\models^M \alpha \).

**2. The logic \( T \) of strict implication.**

Semantic definition of the logic \( T \) of strict implication that is of a consequence \( \models^T \) is analogical to the definition of \( \models^M \). In place of the class \( M \) let us consider the class \( T \) of all Kripke models \( m = <W, r, v, \models_m> \) such
that \( r \) is a reflexive relation on \( W \) and \( W, v, \models_m \) are defined as in a model from the class \( M \).

In order to describe the axiomatic consequence relation of the logic \( T \) let us consider the smallest set \( AxT \) of formulas of the language \( L \) closed on the following rules:

\[
(MP_n) : \alpha_n \to (\alpha_{n-1} \to \ldots \to (\alpha_1 \to (\alpha \to \beta)) \ldots),
\]

\[
\alpha_n \to (\alpha_{n-1} \to \ldots \to (\alpha_1 \to \alpha) \ldots),
\]

\[
\alpha_n \to (\alpha_{n-1} \to \ldots \to (\alpha_1 \to \beta) \ldots), \ n = 0, 1, 2, \ldots
\]

Notice that \((MP_0)\) is of the form: \( \alpha \to \beta, \alpha/\beta \).

Consider the derivability relation \( \vdash RT \) defined by the following set of rules:

\( RT = \{(AxT) \cup \{(R_n) : n = 0, 1, 2 \ldots \} \cup \{(MP_n) : n = 0, 1, 2 \ldots \}\}, \)

where \( (AxT) \) is an axiomatic rule of the form: \( \{<\emptyset, \alpha> : \alpha \in AxT\} \). Then we have

**The soundness theorem:** \( \vdash RT \subseteq \models T \).

For the consequence \( \vdash RT \) we have a counterpart of Lemma 1:

**Lemma 3:** For any theory \( X \) of the logic \( \vdash RT \) and any formulas \( \alpha, \beta : \alpha \to \beta \in X \) iff \( \forall Y(Th(\vdash RT)(<X, Y> \in \varrho \land \alpha \in Y \Rightarrow \beta \in Y)) \), and the relation \( \varrho \), defined as previously, is reflexive on the set \( Th(\vdash RT) \) of all theories of the relation \( \vdash RT \).

**Proof:** The reflexivity of \( \varrho \) follows from the fact that any theory \( X \) of \( \vdash RT \) is closed on \( (MP_0) \). The rest of the lemma is proved similarly as in the proof of Lemma 1.

Applying a notion of canonical model as previously one can prove:

the strong completeness theorem: \( \models T \subseteq \vdash RT \).

### 3. The logic S4 of strict implication.

Consider the consequence \( \models S4 \) determined by the class \( S4 \) of all Kripke models \( m = <W, r, v, \models_m> \) in which the relation \( r \) is reflexive and transitive on \( W \).

In order to describe the axiomatic consequence relation of the logic \( S4 \) of strict implication let us consider the smallest set \( AxS4 \) of formulas closed on the following rules:
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\[(A) : \alpha \to \alpha;\]
\[(Ax1) : (\alpha \to \beta) \to ((\beta \to \gamma) \to (\alpha \to \gamma));\]
\[(Ax2) : (\alpha \to (\beta \to \gamma)) \to ((\alpha \to \beta) \to (\alpha \to \gamma));\]
\[(R) : \alpha/\beta \to \alpha;\]
\[(MP) : \alpha \to \beta, \alpha/\beta.\]

The derivability relation \(\vdash_{RS4}\) is defined by the following set of rules:
\[RS4 = \{ (AxS4), (MP) \}, \text{ where } (AxS4) \text{ is the axiomatic rule of the form:} \]
\[\{ \langle \emptyset, \alpha \rangle : \alpha \in AxS4 \}\].

Then the soundness theorem: \(\vdash_{RS4} \subseteq \models S4\), follows.

As previously one can prove

**Lemma 4:** For any theory \(X\) of the logic \(\vdash_{RS4}\) and any formulas \(\alpha, \beta : \alpha \to \beta \in X\) iff \(\forall Y \in Th(\vdash_{RS4})(\langle X, Y \rangle \in \varrho \text{ and } \alpha \in Y \Rightarrow \beta \in Y)\), where the relation \(\varrho\) defined as previously, is reflexive and transitive on the set \(Th(\vdash_{RS4})\).

**Proof:** One can prove the reflexivity of \(\varrho\) as for the logic \(T\). In order to show that \(\varrho\) is transitive assume that \(\langle X, Y \rangle \in \varrho, \langle Y, Z \rangle \in \varrho, \alpha \to \beta \in X\) and \(\alpha \in Z\), where \(X, Y, Z\) are theories of the logic \(\vdash_{RS4}\). Using the assumption and the fact that \(X\) is closed on \((Ax1)\) and \((MP)\) we get:

\[(\beta \to \beta) \to (\alpha \to \beta) \in X.\]

Since \(Y\) is closed on \((A)\) i.e. \(\beta \to \beta \in Y\) and \(\langle X, Y \rangle \in \varrho\), so from the definition of the relation \(\varrho\) we get \(\alpha \to \beta \in Y\).

From this and the facts that \(\langle Y, Z \rangle \in \varrho, \alpha \in Z\), due to the definition of the relation \(\varrho\), we have immediately that \(\beta \in Z\).

The proof of equivalence of the lemma is similar to the proof of Lemma 1 and Lemma 3. However, there is a difference in pointing out that the set \(Y = \{ \gamma : \alpha \to \gamma \in X\}\), taken into account for the proof of implication \(\alpha \to \beta \notin X\) iff \(\forall Y \in Th(\vdash_{RS4})(\langle X, Y \rangle \in \varrho \text{ and } \alpha \in Y \text{ and } \beta \notin Y)\), is closed on \((MP)\), without using the rules \((MP_n), n = 0, 1...\) So assume that \(\alpha_0 \to \beta_0 \in Y\) and \(\alpha_0 \in Y\). Then we have directly: \(\alpha \to (\alpha_0 \to \beta_0) \in X\) and \(\alpha \to \alpha_0 \in X\). Using \((Ax2)\) and \((MP)\) we get \(\alpha \to \beta_0 \in X\), so \(\beta_0 \in Y\).

Moreover, the last difference consists in applying instead of the rule \((R_0)\) the rules \((Ax1)\) and \((MP)\) in the proof of the fact that \(\langle X, Y \rangle \in \varrho\).

Given Lemma 4 one can construct the canonical model which leads to:

the strong completeness theorem: \(\models S4 \subseteq \vdash_{RS4}\).
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References


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