SOME NUMERICAL CHARACTERIZATION OF FINITE DISTRIBUTIVE LATTICES

Abstract

We regard a finite distributive lattice as the gluing of its maximal boolean intervals and describe some dependencies between dimensions of the boolean intervals and the skeleton of the lattice.

In [7] we described the decomposition of finite lattices into blocks of its skeleton tolerance. In this paper we shall consider the other side of the phenomenon - the gluing of an atlas with overlapping neighbours. Both approaches are equivalent (see [2]).

Let \((L_x)_{x \in K}\) be a family of finite lattices and let the index set \(K\) be also a finite lattice. We call the family \((L_x)_{x \in K}\) a \(K\)-atlas with overlapping neighbours if the following conditions hold for every \(x, y \in K\):

1. If \(L_x \subseteq L_y\) then \(x = y\).
2. If \(x \prec y\) then \(L_x \cap L_y \neq \emptyset\).
3. If \(x \leq y\) and \(L_x \cap L_y \neq \emptyset\) then the orders of \(L_x\) and \(L_y\) coincide on the intersection \(L_x \cap L_y\) and the interval \(L_x \cap L_y\) is at the same time a filter of \(L_x\) and an ideal of \(L_y\).
4. \(L_x \cap L_y = L_x \wedge y \cap L_x \vee y\).

The structure \(L = (\bigcup_{x \in K} L_x, \leq)\), where \(\leq\) is the transitive closure of the union of orders of the lattices \(L_x\) for \(x \in K\), is called the sum of \(K\)-atlas with overlapping neighbours (or simply a \(K\)-gluing of the family \((L_x)_{x \in K}\)).

This definition was introduced by Herrmann in [8].
If \((L_x)_{x \in K}\) is a \(K\)-atlas with overlapping neighbours then the lattices \(L_x\) are intervals of the \(K\)-gluing \(L\). We shall write \(L_x = [0_x, 1_x]\). It is quite natural to consider two mappings \(\sigma, \pi : K \to L\) such that \(\sigma(x) = 0_x\) and \(\pi(x) = 1_x\). It can be proved (see [1])

**Lemma 1.** The mappings \(\sigma\) and \(\pi\) are (strictly) monotone, \(\sigma\) is join-preserving and \(\pi\) is meet-preserving.

It means that

\[
0_x \lor y = 0_x \lor 0_y; \quad (1)
\]

\[
0_x \land y \leq 0_x \land 0_y; \quad (2)
\]

\[
1_x \lor 1_y \leq 1_x \lor y; \quad (3)
\]

\[
1_x \land y = 1_x \land 1_y. \quad (4)
\]

It appears that the operation of gluing of a \(K\)-atlas \((L_x)_{x \in K}\) preserves some important properties of the lattices \(L_x\), regardless of the properties of the lattice \(K\).

Herrmann proved in [8] that if \((M_x)_{x \in K}\) is a \(K\)-atlas with overlapping neighbours and \(M_x\) is a finite modular lattice for every \(x \in K\) then the \(K\)-gluing \(M\) of the \(K\)-atlas is a modular lattice, as well. Similarly, if every lattice \(M_x\) for \(x \in K\) is a distributive one then the \(K\)-gluing \(M\) is also distributive.

Moreover, every finite modular lattice \(M\) is a \(S(M)\)-gluing of its maximal complemented intervals \(M_i\) for \(i \in S(M)\), where \(S(M)\) is a lattice formed by the smallest elements of the lattices \(M_i\) and it is called the skeleton of the lattice \(M\). In the case of finite distributive lattices, complemented lattices coincide with boolean ones and hence every finite distributive lattice \(D\) is the \(S(D)\)-gluing of its maximal boolean intervals.

In fact, the maximal complemented intervals of a finite modular lattice \(M\) are blocks of its skeleton tolerance (i.e. the minimal symmetric and reflexive binary relation on \(M\) which is compatible with the lattice operations and whose transitive closure is the total relation on \(M\).) A block of the skeleton tolerance \(\Sigma(K)\) on a lattice \(K\) is a maximal subset of \(K\) such that every two elements of the subset are in the relation \(\Sigma(K)\) (for details, see for example [7]).
What is more, we can prove (see [6]):

**Lemma 2.** If \((L_x)_{x \in K}\) is a \(K\)-atlas with overlapping neighbours, where \(K\) and \(L_x\) for every \(x \in K\) are finite lattices and \(K_i = [0_i, 1_i]\) for \(i \in S(K)\) are the blocks of the skeleton tolerance \(\Sigma(K)\) then \(x \to 0_x\) and \(x \to 1_x\) are, respectively, the join-embedding of \(K_i\) into \(L_0\) and meet-embedding of \(K_i\) into \(L_1\).

It can be observed (see [5]) that the sizes of the maximal complemented intervals of a finite modular lattices are strictly connected with the shape of its skeleton. In particular, it concerns the dimensions of maximal boolean intervals of finite distributive lattices.

Let \(M\) be a finite modular lattice, \(I\) be an interval of \(M\). We shall denote by \(l(I)\) the length of \(I\), i.e. the length of any maximal chain in \(I\). Let us recall that all maximal chains of a finite modular lattice are of the same length. If \(I\) is a boolean interval then \(l(I) = \dim I\).

**Theorem 3.** Let \((M_x)_{x \in K}\) be a \(K\)-atlas with overlapping neighbours and \(M_x\) be a finite modular lattice for any \(x \in K\). Then for all \(x, y \in K\)

\[
l(M_x) + l(M_y) \leq l(M_{x \lor y}) + l(M_{x \land y}).
\]

**Proof.** Let \(M = \bigcup_{x \in K} M_x\). Then \(M = \langle M, \leq \rangle\), where \(\leq\) is the transitive closure of \(\bigcup_{x \in K} \leq_x\), is a modular lattice being the \(K\)-sum of the family \((M_x)_{x \in K}\). Moreover, \(M_x\) is an interval of \(M\) for every \(x \in K\). Let us denote \(M_x = [0_x, 1_x]\).

Suppose \(l(M_x) = n\), \(l(M_y) = m\) for some \(x, y \in K\). Then there are \(a_1, ..., a_{n-1}, b_1, ..., b_{m-1} \in M\) such that

\[
0_x < a_1 < ... < a_{n-1} < 1_x;
\]

\[
0_y < b_1 < ... < b_{m-1} < 1_y.
\]

By the modularity of \(M\) and by Lemma 1 we get

\[
0_x \lor 0_y \leq 0_x \lor b_1 \leq ... \leq 0_x \lor b_{m-1} \leq ...
\]

\[
\leq 0_x \lor 1_y \leq a_1 \lor 1_y \leq ... \leq 1_x \lor 1_y \leq 1_x \lor 1_y;
\]
\[ 0_{x \land y} \leq 0_x \land 0_y \leq 0_x \land b_1 \leq \ldots \leq 0_x \land b_{m-1} \leq \]
\[ \leq 0_x \land 1_y \leq a_1 \land 1_y \leq \ldots \leq 1_x \land 1_y = 1_{x \land y}. \quad (7) \]

Let us observe that for all \( c_1, c_2, b \in M \), if \( c_1 \prec c_2 \) then it is impossible that the equalities \( c_1 \land b = c_2 \land b \) and \( c_1 \lor b = c_2 \lor b \) hold at the same time. Otherwise, by modularity

\[ c_2 = c_2 \land (b \lor c_2) = c_2 \land (b \lor c_1) = (c_2 \land b) \lor c_1 = (c_1 \land b) \lor c_1 = c_1, \]
which contradicts the assumption on \( c_1 \) and \( c_2 \).

Let us suppose \( l(L_{x \land y}) = k \). If \( k \geq n + m \) then 5 is obvious. Otherwise, in 7 there should be \((n + m - k)\) equalities but, as we observed, every equality in 7 must be accompanied by the strong inequality in 6. Thus

\[ l(L_{x \land y}) + l(L_{x \lor y}) \geq n + m. \quad \bullet \]

**Corollary 4.** Let \( K \) be a finite lattice. For every finite distributive lattice \( D \) with the skeleton \( K \), the maximal boolean intervals \( B_x \) of \( D \) indexed by \( x \in K \) satisfy for all \( x, y \in K \) the condition:

\[ \dim B_x + \dim B_y \leq \dim B_{x \land y} + \dim B_{x \lor y}. \quad (8) \]

We can generalize the above in the following way:

**Theorem 5.** Let \( (B_x)_{x \in K} \) be a \( K \)-atlas with overlapping neighbours and \( B_x \) be a finite boolean lattice for every \( x \in K \). Let \( a, b \) belong to the same block of \( S(K) \). Then, for \( x_1, \ldots, x_n \in K \) \((n \geq 2)\) such that

\[ x_i \land x_j = a; \]
\[ x_i \lor x_j = b \]

for all \( i \neq j, i, j = 1, \ldots, n \), we have

\[ \sum_{i=1}^{n} \dim B_{x_i} \leq \dim B_a + \dim B_b + (n - 2) \dim (B_a \cap B_b). \]

Moreover,

\[ \dim (B_a \cap B_b) \leq \min(\dim B_a, \dim B_b) - n. \]
Proof. For $n = 2$ our Theorem results from Corollary 4. Let us assume $n \geq 3$. It is easy to check that in this case $a < x_i < b$ for every $i = 1, \ldots, n$.

If $a, b$ belong to the same block of $S(K)$ then, by Lemma 2, $0_x \in B_a$, $1_x \in B_b$ for every $i = 1, \ldots, n$ and $0_b \leq 1_a$. Hence every $0_x$, is a meet of some coatoms of $B_a$, i.e. if $\text{CoAt}(B_a) = \{c_1, \ldots, c_k\}$ denotes the set of all coatoms of $B_a$ then

$$0_x = \bigwedge_{s \in I} c_s,$$

for some $I \subseteq \{1, \ldots, k\}$. Let us notice that

$$0_b = 0_{x_i \cup x_j} = 0_{x_i} \lor 0_{x_j} = \bigwedge_{s \in I_i} c_s \lor \bigwedge_{s \in I_j} c_s = \bigwedge_{s \in I_i \cap I_j} c_s. \tag{9}$$

Of course, if $I_i \cap I_j = \emptyset$ then $0_b = \bigwedge \emptyset = 1_a$.

Since 9 holds for every $i \neq j$, where $i, j = 1, \ldots, n$, so there exists $I \subseteq \{1, \ldots, k\}$ such that $I_i \cap I_j = I$ for any $i \neq j$ and then, for every $i = 1, \ldots, n$

$$0_{x_i} = \bigwedge_{s \in I} c_s \land \bigwedge_{s \in I_i \setminus I} c_s = 0_b \land \bigwedge_{s \in I_i \setminus I} c_s.$$

By analogy, if $\text{At}(B_b) = \{a_1, \ldots, a_l\}$ is the set of all atoms of $B_b$ then

$$1_{x_i} = \bigvee_{s \in J} a_s \lor \bigvee_{s \in J \setminus J_i} a_s = 1_a \lor \bigvee_{s \in J \setminus J} a_s,$$

for some $J \subseteq \{1, \ldots, l\}$ such that $J = J_i \cap J_j$ for every $i, j \in \{1, \ldots, n\}$ and $i \neq j$.

Since $[0_b, 1_a] = B_a \cap B_b \neq \emptyset$ and, by the definition of the $K$-atlas with overlapping neighbours, it is a filter of $B_a$ then $[0_b, 1_a]$ is a boolean lattice, hence

$$0_b = \bigwedge_{s \in I} c_s, \text{ where } I \subseteq \{1, \ldots, k\};$$

and

$$1_a = \bigvee_{s \in J} a_s, \text{ where } J \subseteq \{1, \ldots, l\}.$$
Moreover, for every $i = 1, \ldots, n$

$$0_{x_i} < 0_b \leq 1_a < 1_{x_i}$$

and then

$$\dim B_{x_i} = l[0_{x_i}, 0_b] + l[0_b, 1_a] + l[1_a, 1_{x_i}] = |I_i \setminus I| + |I| + |J_i \setminus J|.$$ 

Thus

$$\sum_{i=1}^{n} \dim B_{x_i} = \sum_{i=1}^{n} |I_i \setminus I| + \sum_{i=1}^{n} |J_i \setminus J| + n|I|.$$  

However, since $(I_i \setminus I) \cap (I_j \setminus I) = \emptyset$ for $i \neq j$ then the boolean lattice $B_a$ has at least $\sum_{i=1}^{n} |I_i \setminus I| + |I|$ coatoms, so

$$\dim B_a \geq \sum_{i=1}^{n} |I_i \setminus I| + |I|.$$  

As $0_b < 0_{x_i}$ for every $i = 1, \ldots, n$ then $I_i \setminus I \neq \emptyset$. It means that

$$\dim B_a \geq n + \dim (B_a \cap B_b).$$  

Similarly, $(J_i \setminus J) \cap (J_j \setminus J) = \emptyset$ for $i \neq j$ and

$$\dim B_b \geq \sum_{i=1}^{n} |J_i \setminus J| + |I|,$$

hence

$$\dim B_b \geq n + \dim (B_a \cap B_b).$$

Moreover,

$$\sum_{i=1}^{n} \dim B_{x_i} \leq \dim B_a + \dim B_b + (n - 2) \dim (B_a \cap B_b).$$
References


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