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ON THE DENSITY OF TRUTH IN GRZEGORCZYK'S MODAL LOGIC

Abstract

The paper is an attempt to count the proportion of tautologies of Grzegorczyk's modal calculus among all formulas. We take recourse of some theorems proved in [2].

1. Introduction

Let L be some logical calculus. Let $|T_n|$ be a number of tautologies of length n of that calculus and $|F_n|$ a number of all formulas of that length. We define the density $\mu(L)$ as:

$$\mu(L) = \lim_{n\to\infty} \frac{|T_n|}{|F_n|}$$

The number $\mu(L)$, if it exists, is an asymptotic probability of finding a tautology among all formulas.

In this paper we continue the research concerning the *density of truth* in different logics. Until now, it is known for both classical and intuitionistic logics of implication of one and two variables (see [5], [1]) as well as for the implicational-negational fragments of those logics with one variable (see [8], [3], [4]).

In this note we intend to estimate the *density of truth* for Grzegorczyk's logic and give this logic exact value for some normal extension of it.

2. Grzegorczyk's logic and its normal extensions

Syntactically, Grzegorczyk's logic **Grz** is characterized as a normal extension of **S4** modal calculus by the axiom

$$(grz) \ \ \Box(\Box(p \to \Box p) \to p) \to p$$

The set of rules consists of modus ponens, substitution and necessitation.

Since the problem of density within Grzegorczyk's logic is a complex one, we have to confine ourselves to the language $\mathcal{F}^{\{\to,\Box\}}$ composed only of sings of implication and necessity and one propositional variable p only. Its formal definition is given in [2].

We will consider logics $\mathbf{Grz}^{\leq n} = \mathbf{Grz} \oplus J_n$ (see [2]), containing the logic **Grz** and satisfying the following inclusions:

$$\mathbf{Grz} \subset ... \subset \mathbf{Grz}^{\leq n} \subset \mathbf{Grz}^{\leq n-1} \subset ... \subset \mathbf{Grz}^{\leq 2} \subset \mathbf{Grz}^{\leq 1} \tag{1}$$

3. Counting formulas and generating functions

In this section we set up the method of counting formulas of the established length. We will consider the set $F_n \subseteq \mathcal{F}^{\{\to,\Box\}}$ of all formulas of length n. The way of measuring the length of formula has been determined in [2], Definition 9.

Definition 1. By F_n we mean the set of formulas from $\mathcal{F}^{\{\to,\Box\}}$ of the length $n-1$.

We will also consider some appropriate subclasses of F_n. For example, if we have a class $A \in \mathcal{F}^{\{\to,\Box\}}$ then $A_n = F_n \cap A$ and

Definition 2. By $|A_n|$ we mean the number of formulas from the class A_n.

Lemma 3. *The number $|F_n|$ of formulas from F_n is given by recursion:*

$$|F_0| = |F_1| = 0, \quad |F_2| = 1, \tag{2}$$

$$|F_n| = |F_{n-1}| + \sum_{i=1}^{n-2} |F_i||F_{n-i}|. \tag{3}$$

Proof. Any formula of length $n-1$ for $n > 2$ is either a necessitation of some formula of length $n-2$ to which the fragment $|F_{n-1}|$ corresponds, or an implication of formulas of lengths $i-1$ and $n-i-1$, respectively. The length of any implicational formula must be $(i-1)+(n-i-1)+1$ which yields exactly $n-1$. Therefore the total number of such formulas is $\sum_{i=1}^{n-2} |F_i||F_{n-i}|$. □

In case of asymptotics of sequences of numbers, *generating functions* (see for example [7]) are the best tool. Let $A = (A_0, A_1, A_2, \ldots)$ be a sequence of real numbers. It is in one-to-one correspondence to the formal power series $\sum_{n=0}^{\infty} A_n z^n$. Moreover, considering z as a complex variable, this series converges uniformly to a function $f_A(z)$ in some open disc $\{z \in \mathcal{C} : |z| < R\}$. So, with the sequence A we can associate a complex function $f_A(z)$, called the *ordinary generating function* for A, defined in a neighborhood of 0. This correspondence is one-to-one again (unless $R = 0$), since the expansion of a complex function $f(z)$, analytic in a neighborhood of z_0, into a power series $\sum_{n=0}^{\infty} A_n (z - z_0)^n$ is unique, and moreover, this series is the Taylor series, given by

$$A_n = \frac{1}{n!} \frac{d^n f}{dz^n}(z_0). \tag{4}$$

Many questions concerning the asymptotic behaviour of A can be efficiently resolved by analyzing the behaviour of f_A at the complex circle $|z| = R$.

The key tool will be the following result due to Szegö [6], Thm. 8.4, see also [7], Thm. 5.3.2, which relates the generating functions of numerical sequences to the limit of the fractions being investigated. For the technique of proof described below please consult also [5] as well as [8]. We need the following, much simpler, version of the Szegö lemma.

Lemma 4. *Let $v(z)$ be analytic in $|z| < 1$ with $z = 1$ being the only singularity at the circle $|z| = 1$. If $v(z)$ in the vicinity of $z = 1$ has an expansion of the form*

$$v(z) = \sum_{p \geq 0} v_p (1 - z)^{\frac{p}{2}}, \tag{5}$$

where $p > 0$, and the branch chosen above for the expansion equals $v(0)$ for $z = 0$, then

$$[z^n]\{v(z)\} = v_1 \binom{1/2}{n} (-1)^n + O(n^{-2}). \tag{6}$$

The symbol $[z^n]\{v(z)\}$ stands for the coefficient of z^n in the exponential series expansion of $v(z)$.

First, we determine the generating function for the sequence of numbers $|F_n|$.

Lemma 5. *The generating function f_F for the numbers $|F_n|$ is*

$$f_F(z) = \frac{1 - z}{2} - \frac{\sqrt{(z + 1)(1 - 3z)}}{2}. \tag{7}$$

Proof. The recurrence $|F_n| = |F_{n-1}| + \sum_{i=1}^{n-2} |F_i||F_{n-i}|$ becomes the equality

$$f_F(z) = z f_F(z) + f_F^2(z) + z^2 \tag{8}$$

since the recursion fragment $\sum_{i=1}^{n-2} |F_i||F_{n-i}|$ corresponds exactly with multiplication of power series. The term $|F_{n-1}|$ corresponds with the function $z f_F(z)$. The quadratic term z^2 corresponds with the first non-zero coefficient in the power series of f_F. Solving the equation we get two possible solutions: $f_F(z) = (1 - z)/2 - \sqrt{-3z^2 - 2z + 1}/2$ or $f_F(z) = (1 - z)/2 + \sqrt{-3z^2 - 2z + 1}/2$. We have to choose the first one, since it corresponds to the assumption $f_F(0) = 0$ (see equation (2)). □

4. Upper estimation of the *density*

In this section we count the density of the logic $\mathbf{Grz}^{\leq 2}$ (for details see [2]). Since the inclusions (1) hold, we conclude that

$$\mu(\mathbf{Grz}) < \mu(\mathbf{Grz}^{\leq n})$$

for every $n \in \mathsf{N}$.

It would be desirable to count the *density* of $\mathbf{Grz}^{\leq n}$ for any $n \in \mathsf{N}$, but we have not been able to do this. Unfortunately, even for $n = 3$ the needed calculations are extremely complicated. We manage to count the density for $n = 2$.

To simplify the notation instead "the quotient algebra $\mathbf{Grz}^{\leq 2}/_{\equiv}$" we write "$AL$". AL is presented below in the Diagram 1.

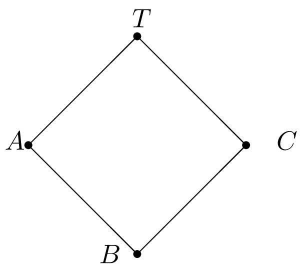

Diagram 1

where

$$A = [p]_{\equiv}, \quad B = [\Box p]_{\equiv}, \quad C = [p \to \Box p]_{\equiv}, \quad T = [p \to p]_{\equiv}$$

Observation 6. *The operations* $\{\to, \Box\}$ *in the algebra* AL *can be displayed by the following truth table:*

$\to$	A	B	C	T	$\Box$
A	T	C	C	T	B
B	T	T	T	T	B
C	A	A	T	T	C
T	A	B	C	T	T

Table 1.

For technical reason we are going to consider a new algebra obtained from the above one by an appropriate identification. We take the open filter $[C)$. Let us consider the algebra $AL_1 = AL/_{[C)}$. It is easy to observe that $AL_1 = \mathbf{Grz}^{\leq 1}/_{\equiv}$ and its diagram is the following:

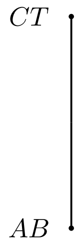

Diagram 2

where

$$AB = A \cup B, \quad CT = C \cup T$$

OBSERVATION 7. *The operations* $\{\to, \Box\}$ *in the algebra* AL_1 *are given by the following truth table:*

$\to$	AB	CT	$\Box$
AB	CT	CT	AB
CT	AB	CT	CT

Table 2.

Now, we determine the generating function f_T for the class T of tautologies of $\mathbf{Grz}^{\leq 2}$. To do that we start with calculating the generating functions f_{AB}, f_{CT} and f_C.

LEMMA 8. *The generating function* f_{AB} *for the numbers* $|AB_n|$ *is*

$$f_{AB}(z) = \frac{f(z) - 1 + z + X}{2}. \qquad (9)$$
$$\text{where} \quad X = \sqrt{4z^2 + z(f(z) - 2) - f(z) + 1}$$

For the sake of simplicity we have written this function in the terms of function f.

PROOF. Table 2 shows that any formula from the class AB of length $n-1$ is either a necessitation of formula from the same class AB of length $n-2$ or an implication of formulas from classes CT and AB of length $i-1$ and $n-i-1$, respectively. We also know that $p \in AB$. That gives the recurrence

$$|AB_0| = |AB_1| = 0, \;\; |AB_2| = 1, \tag{10}$$

$$|AB_n| = |AB_{n-1}| + \sum_{i=1}^{n-1} |CT_i||AB_{n-i}| \tag{11}$$

From disjointness of classes AB and CT we have $|CT_i| = |F_i| - |AB_i|$. Hence $|AB_n| = |AB_{n-1}| + \sum_{i=1}^{n-1}(|F_i| - |AB_i|)|AB_{n-i}|$.

The number $|AB_{n-1}|$ corresponds to the function $zf_{AB}(z)$. The quadratic term z^2 corresponds to the first non-zero coefficient in the power series of f_{AB}. The recursion fragment $\sum_{i=1}^{n-2}(|F_i| - |AB_i|)|AB_{n-i}|$ corresponds exactly to the multiplication of power series. Hence we have the equation:

$$f_{AB}(z) = (f(z) - f_{AB}(z))f_{AB}(z) + zf_{AB}(z) + z^2. \tag{12}$$

By solving it with the boundary condition $f_{AB}(0) = 0$ we have (9).

□

COROLLARY 9. *The generating function f_{CT} for the numbers $|CT_n|$ is*

$$f_{CT}(z) = \frac{f(z) + 1 - z - X}{2}. \quad \textit{where} \quad X = \sqrt{4z^2 + z(f(z) - 2) - f(z) + 1} \tag{13}$$

PROOF. It follows from disjointness of classes AB and CT that $f_{CT} = f - f_{AB}$.

□

LEMMA 10. *The generating function f_C for the numbers $|C_n|$ is*

$$f_C(z) = \frac{1}{6}\left(2^{\frac{2}{3}}Y - \frac{2^{\frac{4}{3}}U}{Y} - X - z + 3(f-1)\right) \tag{14}$$

where

$$\begin{aligned}
Y &= \sqrt[3]{S+\sqrt{4U^3+S^2}},\\
S &= \frac{1}{2}\left(X(19z^2+2z(11f-13)-4(f-1))+43z^3+3z^2(7f-17)+\right.\\
&\left.30z(1-f)\right),\\
U &= -\frac{1}{2}\left(zX+z^2-z(f+1)-2(f-1)\right),\\
X &= \sqrt{4z^2+z(f-2)-f+1}
\end{aligned}$$

For simplicity we have omitted in the above function the argument (z) *and have written* f *instead of* $f(z)$. *We shell do the same hereafter.*

Proof. From Table 1 we can notice that the following recurrence for the numbers $|B_n|$ holds:

$$\begin{aligned}
|B_0| &= 0, \quad |B_1| = 0\\
|B_n| &= (|A_{n-1}|+|B_{n-1}|)+\sum_{i=1}^{n-1}|T_i||B_{n-i}| \qquad (15)
\end{aligned}$$

This can be translated into equation:

$$f_B = f_T f_B + (f_A+f_B)z. \qquad (16)$$

Since $f_A+f_B = f_{AB}$ and $f_T = f_{CT} - f_C$ then we have:

$$f_B = \frac{zf_{AB}}{1-f_{CT}+f_C}. \qquad (17)$$

Table 1 suggests also that the recursion schema for the class C must be:

$$\begin{aligned}
|C_0| &= 0, \quad |C_1| = 0\\
|C_n| &= |C_{n-1}|+\sum_{i=1}^{n-1}(|A_i|(|B_{n-i}|+|C_{n-i}|)+|T_i||C_{n-i}|) \qquad (18)
\end{aligned}$$

The above recurrence gives the following equality between generating functions:

$$f_C = zf_C + (f_B+f_C)f_A + f_T f_C \qquad (19)$$

The unknown functions from (19) can be replaced by those already known. We know that $f_A = f_{AB} - f_B$ and $f_T = f_{CT} - f_C$. After application of the above equalities to the (19) we get

$$f_C \quad = \quad z f_C + ((f_{AB} - f_A + f_C)(f_{AB} - f_B) + (f_{CT} - f_C) f_C \tag{20}$$

From the system of equations

$$\begin{cases} (17) \\ (20) \end{cases}$$

we have obtained a four-degree equation with the unknown function f_C. To solve it we have to intensively use the *Mathematica* package and from four solutions we choose one satisfying the boundary condition $f_C(0) = 0$. Then we have (14) presenting the function f_C in terms of some expressions Y, S, U, X. □

COROLLARY 11. *The generating function f_T for the numbers $|T_n|$ is*

$$f_T = f_{CT} - f_C \tag{21}$$

where the functions f_{CT} and f_C are defined by (13) and (14).

To apply the Szegö lemma we need functions which are analytic in the open disc $|z| < 1$, and the nearest singularity is at $z_0 = 1$. For that purpose we are going to calibrate functions f and f_T in the following way:

$$\begin{aligned} \widehat{f}(z) &= f\left(\tfrac{z}{3}\right) & \qquad \widehat{f_{CT}}(z) &= f_{CT}\left(\tfrac{z}{3}\right) \\ \widehat{f_C}(z) &= f_C\left(\tfrac{z}{3}\right) & \qquad \widehat{f_T}(z) &= f_T\left(\tfrac{z}{3}\right). \end{aligned}$$

After the appropriate simplification of the above expressions we obtain the following:

$$\widehat{f}(z) = \frac{1}{6}\left(3 - z - \sqrt{3}\sqrt{(z+3)(1-z)}\right) \tag{22}$$

$$\widehat{f_{CT}}(z) = \frac{3\widehat{f} + 3 - z - \widehat{X}}{6} \tag{23}$$

$$\widehat{f_C}(z) = \frac{(2^{\frac{2}{3}}\widehat{Y} - \frac{2^{\frac{4}{3}}\widehat{U}}{\widehat{Y}} - \widehat{X} - z + 9(\widehat{f} - 1)}{18} \tag{24}$$

$$\widehat{f_T} = \widehat{f_{CT}} - \widehat{f_C} \tag{25}$$

where

$$\begin{aligned}
\widehat{Y} &= \sqrt[3]{\widehat{S} + \sqrt{4\widehat{U}^3 + \widehat{S}^2}}, \\
\widehat{S} &= \frac{1}{54}\Big(3\widehat{X}(19z^2 + 6z(11\widehat{f} - 13) - 36(\widehat{f} - 1)) + 43z^3 + \\
&\quad 9z^2(7\widehat{f} - 17) + 270z(1 - \widehat{f})\Big), \\
\widehat{U} &= -\frac{1}{18}\Big(3z\widehat{X} + z^2 - 3z(\widehat{f} + 1) - 18(\widehat{f} - 1)\Big), \\
\widehat{X} &= \frac{1}{3}\sqrt{4z^2 + 3z(\widehat{f} - 2) - 9\widehat{f} + 9}
\end{aligned}$$

Note that relations between power series of appropriate functions are such as $[z^n]\{f(z)\} = \Big([z^n]\{\widehat{f}(z)\}\Big)\,3^n$.

Lemma 12. *$z_0 = 1$ is the only singularity of $\widehat{f}$ and $\widehat{f_T}$ located in $|z| \leq 1$.*

Proof. It is easy to observe that the function $\widehat{f}(z)$ has only singularities at $z = 1$ and $z = -3$. To make sure that the function $\widehat{f_T}(z)$ has the nearest singularity at $z = 1$, we have to solve the following complicated equations:

$$\begin{aligned}
\widehat{X} &= 0 \\
\widehat{Y} &= 0 \\
4\widehat{U}^3 + \widehat{S}^2 &= 0
\end{aligned}$$

To this aim we have to extensively use *the Mathematica* package. It occurrs that all solutions which are different from $z = 1$ are situated outside the disc $|z| \leq 1$.

Theorem 13. *Expansions of functions $\widehat{f}$ and $\widehat{f_T}$ in a neighborhood of $z = 1$ are as follows:*

$$\widehat{f}(z) = f_0 + f_1\sqrt{1 - z} + ...$$
$$\widehat{f_T}(z) = t_0 + t_1\sqrt{1 - z} + ...$$

where

$$f_0 = \frac{1}{3}, \quad f_1 = -\frac{1}{\sqrt{3}}, \quad \ldots, \quad t_0 = 0.104415..., \quad t_1 \quad = \quad -0.356051...$$

PROOF. The above coefficients have been found using *the Mathematica* package. The exact values of the coefficients t_0 and t_1 are too long to be written here. □

Now, we can calculate the density of implicational-necessitional part of the extension of Grzegorczyk's logic $\mathbf{Grz}^{\leq 2}$ of one variable. By applying the Szegö lemma we get as follows:

THEOREM 14.

$$\begin{aligned}\mu(\mathbf{Grz}^{\leq 2}) &= \lim_{n\to\infty} \frac{|T_n|}{|F_n|} = \lim_{n\to\infty} \frac{(t_1\binom{1/2}{n}(-1)^n + O(n^{-2}))3^n}{(f_1\binom{1/2}{n}(-1)^n + O(n^{-2}))3^n} \\ &= \lim_{n\to\infty} \frac{t_1}{f_1}(1+o(1)) = \frac{t_1}{f_1} \approx 61.27\%\end{aligned}$$

5. Lower estimation of the *density*

DEFINITION 15. The set of simple modal tautologies is defined as follows:

1. $p \to p \in ST$,
2. $\Box(\Box(p \to \Box p) \to p) \to p \in ST$,
3. If $\alpha \in ST$ then $\Box\alpha \in ST$,
4. If $\alpha \in ST$ then $\beta \to \alpha \in ST$ for every $\beta \in \mathcal{F}^{\{\to,\Box\}}$,
5. If $\alpha \notin ST$, then $\underbrace{\Box ... \Box}_{k-times} p \to \alpha \in ST$ for $k \geq 1$.

From the above definition it is easy to notice that the set of simple tautologies is a proper subset of the set of the ones of Grzegorczyk's logic. Hence we have:

OBSERVATION 16. $\mu(ST) < \mu(\mathbf{Grz})$

LEMMA 17. *The numbers $|ST_n|$ of formulas from ST_n are given by the recursion:*

$$|ST_0| \quad = \quad ... = |ST_3| = 0, \ \ |ST_4| = 1, \tag{26}$$

$$|ST_n| = |ST_{n-1}| + \sum_{i=1}^{n-2} |F_{n-i}||ST_i| + \underbrace{((|F_{n-3}| - |ST_{n-3}|) + (|F_{n-4}| - |ST_{n-4}|) + ... + (|F_2| - |ST_2|))}_{(n-4)-times}. \quad (27)$$

PROOF. From Definition 15 we see that the simple modal tautologies of the length $n-1$ are either a necessitation of a simple modal tautology of the length $n-2$ or an implication of some pairs consisting of a formula and a simple modal tautology or the formula $\underbrace{\Box ... \Box}_{k-times} p$ and a formula which is not a simple modal tautology. □

LEMMA 18. *The generating function f_{ST} for the numbers $|ST_n|$ is the following:*

$$f_{ST}(z) = \frac{z^4 + z^{11} + \frac{fz^3(1-z^{-4+n})}{1-z}}{1 - f - z + \frac{z^3(1-z^{-4+n})}{1-z}} \quad (28)$$

PROOF. From the recurrence (27) we obtain the generating function f_{ST} must satisfy the following equation:

$$f_{ST}(z) = f_{ST}(z)z + f(z)f_{ST}(z) + (f(z) - f_{ST}(z))(z^3 + z^4 + ... + z^{n-2}) + z^4 + z^{11} \quad (29)$$

Since $z^3 + z^4 + ... + z^{n-2} = z^3\frac{1-z^{n-4}}{1-z}$ then after solving (29) with the boundary condition $f_{ST}(0) = 0$ we get (28). □

As in the previous section we calibrate the function f_{ST}:

DEFINITION 19. $\widehat{f_{ST}}(z) = f_{ST}(\frac{z}{3})$.

After a suitable substitution we have:

$$\widehat{f_{ST}}(z) = \frac{\left(\frac{z}{3}\right)^4 + \left(\frac{z}{3}\right)^{11} + \frac{fz^3(1-3^{4-n}z^{-4+n})}{9(3-z)}}{1 - f - \frac{z}{3} + \frac{z^3(1-3^{4-n}z^{-4+n})}{9(3-z)}} \quad (30)$$

Now, we should check that the only singularity situated in disc $|z| \leq 1$ of the function (28) is the point $z = 1$. We set up that $n = 10$ (which has no significant influence upon our calculations) and obtain:

$$\widehat{f_{ST}}^{*}(z) \quad = \quad \frac{\left(\frac{z}{3}\right)^4 + \left(\frac{z}{3}\right)^{11} + \frac{fz^3(1-3^{-6}z^6)}{9(3-z)}}{1 - f - \frac{z}{3} + \frac{z^3(1-3^{-6}z^6)}{9(3-z)}} \tag{31}$$

LEMMA 20. *$z_0 = 1$ is the only singularity of the function $\widehat{f_{ST}}^{*}$ located in $|z| \leq 1$.*

PROOF. We check that the following equation has no solution at the disc $|z| \leq 1$:

$$1 - f - \frac{z}{3} + \frac{z^3(1-3^{-6}z^6)}{9(3-z)} \quad = \quad 0$$

We have used *the Mathematica* package. □

THEOREM 21. *Expansion of function $\widehat{f_{ST}}^{*}$ in a neighborhood of $z = 1$ is as follows:*

$$\widehat{f_{ST}}^{*}(z) = t_0^* + t_1^*\sqrt{1-z} + ...$$

where

$$t_0^* = \frac{5464}{68877}, \quad t_1^* = -\frac{2256316\sqrt{3}}{19522803}, ...$$

PROOF. The above coefficients have been found using *the Mathematica* package. □

Now, we have the value of the *density* of the set of simple modal tautologies:

THEOREM 22.

$$\begin{aligned}\mu(ST) \quad &= \quad \lim_{n\to\infty} \frac{|ST_n|}{|F_n|} = \lim_{n\to\infty} \frac{(t_1^*\binom{1/2}{n}(-1)^n + O(n^{-2}))3^n}{(f_1\binom{1/2}{n}(-1)^n + O(n^{-2}))3^n} \\ &= \quad \lim_{n\to\infty} \frac{t_1^*}{f_1}(1 + o(1)) = \frac{t_1^*}{f_1} \approx 34.67\%\end{aligned}$$

Theorems 14 and 22 give us some information about the *density* of implicational-necessitational fragment of Grzegorczyk's logic of one variable. We know only that:

$$34.67\% < \mu(\mathbf{Grz}) < 61.27\% \tag{32}$$

Since the method of counting the *densities* of $\mathbf{Grz}^{\leq 3}$ is the same as the one of $\mathbf{Grz}^{\leq 2}$ (see Diagram 3 in [2]), we hope that the inequalities will be soon improved, especially, the upper estimation. The only problem in that case concerns degrees of complexity of some equations.

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