FORMALIZATION OF A PLAUSIBLE INference

The key intuition behind plausibility in the sense of Ajdukiewicz [1] is that it admits reasonings wherein the degree of strength of the conclusion (i.e. the conviction it is true) is smaller than that of the premises. In consequence in the formal description of plausible inference, the standard notions of consequence and rule of inference cannot be countenanced as entirely suitable.

Accordingly, for a formal description of plausible inference, the notions of rule of inference and consequence adequate for the deductive case are not fully appropriate.

The main difference between the deductive and the plausible inferences lies in the fact that the condition:

\[ (* ) \text{ if } X \vdash \alpha_1, \ldots, X \vdash \alpha_n \text{ and } \{\alpha_1, \ldots, \alpha_n\} \vdash \alpha \text{ then } X \vdash \alpha. \]

holds for the former but is dispensable for the latter. In terms of the operations on the set of formulas, this means that the condition \[ C(C(X)) \subseteq C(X) \]

may not hold for a proper plausible consequence relation. Notwithstanding, nothing prevents us from accepting other Tarski conditions, the notion of plausible consequence being a generalization of Tarskian consequence operation.

On the other hand, the notion of \( p \)-consequence operation turns out to be closely related to the so-called quasi-consequence (\( q \)-consequence) operation considered by Malinowski [4]. Single matrices for the operations of both types contain two sets of designated values: one of possible values (degrees of truth) for the premisses, the other for the conclusion. In a matrix determining a \( p \)-consequence operation, the set of possible values
for a conclusion is included in the set of the values for premisses (contrary to the q-consequence case); the values from the smaller set represent the higher degrees of truth, yet the premisses might be justified better than a conclusion in a weak-deductive reasoning.

1. \textit{p-consequence operation, p-derivability relation}

**Definition 1.** By a \textit{p-consequence operation} for a sentential language \( \mathcal{L} \) we mean any function \( Z : 2^L \rightarrow 2^L \) that is subject to the conditions (for all \( X \subseteq L, \alpha \in L \)):

(i) \( X \subseteq Z(X) \);

(ii) \( Z(X) \subseteq Z(Y) \) whenever \( X \subseteq Y \).

Moreover, if a \textit{p-consequence operation} \( Z \) fulfils the condition

(iii) \( Z(X) = \bigcup \{ Z(X_f) : X_f \in \text{Fin}, X_f \subseteq X \} \), (\( \text{Fin} \) is the family of all finite sets of formulas), then \( Z \) will be called \textit{finitary}.

We can additionally define the property of structurality for \( p \)-consequence \( Z \): \( Z \) is \textit{structural} iff \( eZ(X) \subseteq Z(eX) \) for every endomorphism \( e \in \text{End}(\mathcal{L}) \).

Obviously, any consequence operation is a \( p \)-consequence.

**Definition 2.** By a \textit{p-inference for the language} \( \mathcal{L} \) we shall understand any finite sequence \((a_1, \ldots, a_n)\), \( n \geq 1 \), of ordered pairs from the Cartesian product \( L \times \{ *, 1 \} \). By a \textit{p-rule of inference} for the language \( \mathcal{L} \) we mean an arbitrary nonempty set of \( p \)-inferences for \( \mathcal{L} \).

For example, the sequence \(( < p \rightarrow q, * >, < p, 1 >, < q, * > )\) is a \( p \)-inference while the set \( \{ < (\alpha \rightarrow \beta, x_1 >, < \alpha, x_2 >, < \beta, * > ) : \alpha, \beta \in L, x_1, x_2 \in \{ *, 1 \}, x_1 = 1 \text{ or } x_2 = 1 \} \) is a \( p \)-rule of inference.

**Definition 3.** Given the language \( \mathcal{L} = (L, f_1, \ldots, f_n) \) by a \textit{p-proof} of a formula \( \alpha \) from a set \( X \) of formulas based on a set \( \mathcal{R} \) of \( p \)-rules of inference is a \( p \)-inference \((a_1, \ldots, a_k), k \geq 1 \), for \( \mathcal{L} \) such that
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(i) \( pr_1(a_k) = \alpha \);
(ii) for all \( i = 1, 2, \ldots, k \), either \( pr_1(a_i) \in X \) and \( pr_2(a_i) = 1 \) or there exists a \( p \)-rule \( r \in R \) and a \( p \)-inference \( (b_1, \ldots, b_j) \in r \) such that \( a_i = b_j \) and \( \{b_1, \ldots, b_{j-1}\} \subseteq \{a_1, \ldots, a_{i-1}\} \). \( (pr_1, pr_2 \) are the first and the second projections on \( L \times \{*, 1\} \), respectively).

We say that a formula \( \alpha \) is \( p \)-derivable from a set of formulae \( X \) by the \( p \)-rules from \( R \) (\( X \models_R \alpha \) in symbols) iff there is a \( p \)-proof of \( \alpha \) from \( X \) on the basis of \( R \). The relation \( \models_R \) will be called a \( p \)-derivability relation determined by the set of \( p \)-rules \( R \). We shall write \( (a_1, \ldots, a_k, < \beta, x_1>, \ldots, < \beta, x_m >, c_1, \ldots, c_n) \) instead \( (a_1, \ldots, a_k, < \beta, x_1 >, \ldots, < \beta, x_m >, c_1, \ldots, c_n) \).

Let \( (a_1, \ldots, a_n) \) be \( p \)-inference. Let for each \( i \in \{1, \ldots, n\} : A_i(i) = \{pr_1(a_i) : 1 \leq l \leq i \& pr_2(a_i) = *) \) and \( A_1(i) = \{pr_1(a_i) : 1 \leq l \leq i \& pr_2(a_i) = 1 \} \). Now for any \( p \)-consequence \( Z \) on the language \( L \) let us distinguish the following set \( \mathcal{R}(Z) \) of \( p \)-rules of inference:

\[ r \in \mathcal{R}(Z) \text{ iff for any } Y \subseteq L \text{ and } p \text{-inference } (a_1, \ldots, a_n) \in r, \text{ the conditions: } A_n(n - 1) \subseteq Z(Y), Z(Y \cup A_1(n - 1)) = Z(Y) \text{ imply that } \langle pr_2(a_n) = * \Rightarrow pr_1(a_n) \in Z(Y) \rangle \&(pr_2(a_n) = 1 \Rightarrow Z(Y, pr_1(a_n)) = Z(Y)) \]

If \( Z \) is finitary, it coincides with the \( p \)-derivability relation determined by the set \( \mathcal{R}(Z) \):

**Theorem 1.** For any finitary \( p \)-consequence \( Z \) on the language \( L \), any \( X \subseteq L \) and \( \alpha \in L : \alpha \in Z(X) \) iff \( X \models_{\mathcal{R}(Z)} \alpha \).

**Proof:** \( \Rightarrow \): Assume first that \( \alpha \in Z(X) \). Thus \( \alpha \in Z(a_1, \ldots, a_{n-1}) \), where \( \{a_1, \ldots, a_{n-1}\} \subseteq X \) for \( Z \) being finitary. Consider 1-element \( p \)-rule of inference: \( r_0 = \{< \alpha, 1 >, < \alpha, * >\} \). Then the \( p \)-inference \( \langle < \alpha, 1 >, < \alpha, * >\rangle \) is a \( p \)-proof of \( \alpha \) from \( X \) on the basis of \( \mathcal{R}(Z) \), where the last element of the proof is obtained by \( r_0 \). Moreover – obviously \( r_0 \in \mathcal{R}(Z) \).

\( \Rightarrow \): Assume that \( X \models_{\mathcal{R}(Z)} \alpha \). It follows that there is a \( p \)-proof \( (a_1, \ldots, a_k) \) of \( \alpha \) from \( X \) on the basis of \( \mathcal{R}(Z) \) such that \( pr_1(a_k) = \alpha \). We have to show that \( pr_1(a_k) \in Z(X) \). It is sufficient to prove by induction that for all \( i \in \{1, \ldots, k\} \):

\( * \quad A_i(i) \subseteq Z(X) \& Z(X \cup A_1(i)) = Z(X) \)
First we show that (*) holds for \( i = 1 \). Indeed, we have: either (a) \( pr_1(a_1) \in X \) or (b) there exists \( r \in R(Z) \) such that \( (a_1) \in r \). If (a) holds then \( A_* (1) = \emptyset \) and moreover, \( Z(X \cup A_1(1)) = Z(X) \) for \( A_1(1) = \{ pr_1(a_1) \} \). If (b) holds then \( r \) from the definition of the set \( R(Z) \) applied for the set \( X \) and \( p \)-inference \((a_1)\), it follows that \( (pr_2(a_1)) = * \Rightarrow pr_1(a_1) \in Z(X) \) & \( (pr_2(a_1)) = 1 \Rightarrow Z(X, pr_1(a_1)) = Z(X) \). If \( pr_2(a_1) = * \), then \( A_* (1) = \{ pr_1(a_1) \} \subseteq Z(X) \) and \( A_1(1) = \emptyset \), and if \( pr_2(a_1) = 1 \), then \( A_* (1) = \emptyset \) and \( Z(X \cup A_1(1)) = Z(X) \) for \( A_1(1) = \{ pr_1(a_1) \} \).

Now assume that (*) holds for \( i \in \{ 1, \ldots, k \} \). To show that it also holds for \( i + 1 \), consider the disjunction: (c) \( pr_1(a_{i+1}) \in X \) or (d) for some \( r \in R(Z) \) there is a \( p \)-inference \((b_1, \ldots, b_m, a_{i+1}) \in r \) such that

1. \( \{ b_1, \ldots, b_m \} \subseteq \{ a_1, \ldots, a_i \} \).

If (c) holds, then \( A_*(i + 1) = A_*(i) \), therefore \( A_*(i + 1) \subseteq Z(X) \) by inductive assumption; moreover, \( A_1(i + 1) = A_1(i) \cup \{ pr_1(a_{i+1}) \} \), so from the inductive assumption it follows that \( Z(X \cup A_1(i + 1)) = Z(X) \).

If (d) holds, first let us put for any \( j \in \{ 1, \ldots, m \} : B_*(j) = \{ pr_1(b_j) : 1 \leq l \leq j \text{ and } (pr_2(b_j) = *) \} \) and \( B_1(j) = \{ pr_1(b_j) : 1 \leq l \leq j \text{ and } (pr_2(b_j) = 1) \} \).

By the definition of the set \( R(Z) \) follows that

2. \( B_*(m) \subseteq Z(X \cup A_1(i)) \) & \( Z(X \cup A_1(i) \cup B_1(m)) = Z(X \cup A_1(i)) \).

Since, as a result of (1) it follows that \( B_*(m) \subseteq A_*(i) \) and \( B_1(m) \subseteq A_1(i) \), then by the inductive assumption: \( B_*(m) \subseteq Z(X) \subseteq Z(X \cup A_1(i)) \) & \( Z(X \cup A_1(i) \cup B_1(m)) = Z(X \cup A_1(i)) \). Therefore from (2):

3. \( pr_2(a_{i+1}) = * \Rightarrow pr_1(a_{i+1}) \in Z(X \cup A_1(i)) \) and
4. \( pr_2(a_{i+1}) = 1 \Rightarrow Z(X \cup A_1(i), pr_1(a_{i+1})) = Z(X \cup A_1(i)) \).

In case \( pr_2(a_{i+1}) = * : A_* (i + 1) = A_*(i) \cup \{ pr_1(a_{i+1}) \} \). However by inductive assumption, \( A_*(i) \subseteq Z(X) \), and from (3): \( pr_1(a_{i+1}) \in Z(X \cup A_1(i)) = Z(X) \), thus \( A_*(i + 1) \subseteq Z(X) \). Moreover, \( A_1(i + 1) = A_1(i) \) so \( Z(X \cup A_1(i + 1)) = Z(X) \) by the inductive assumption. In case \( pr_2(a_{i+1}) = 1 : A_* (i + 1) = A_*(i) \subseteq Z(X) \), and \( A_1(i + 1) = A_1(i) \cup \{ pr_1(a_{i+1}) \} \). So \( Z(X \cup A_1(i + 1)) = Z(X \cup A_1(i), pr_1(a_{i+1})) = Z(X \cup A_1(i)) = Z(X) \) due to (4) and the inductive assumption. \( \square \)
2. \textit{p-matrices}

Let $\mathcal{L} = (L, f_1, \ldots, f_n)$ be a sentential language and let $\mathcal{M} = (M, F_1, \ldots, F_n, D_1, D_2)$ be a relational structure such that $M = (M, F_1, \ldots, F_n) \in \mathcal{L}$ is an algebra similar to $\mathcal{L}$ and $D_1, D_2$ are sets such that $\emptyset \neq D_1 \subseteq D_2 \subseteq M$. Call such a structure $\mathcal{M}$ a \textit{p-matrix} for the language $\mathcal{L}$. Let us define for any p-matrix $\mathcal{M}$ the following operation $Z_M : 2^L \rightarrow 2^L$.

**Definition 4.** For every $X \subseteq L$, $\alpha \in L$:

$\alpha \in Z_M(X)$ if and only if $\forall h \in \text{Hom}(\mathcal{L}, \mathcal{M})[h(X) \subseteq D_1 \Rightarrow h\alpha \in D_2]$.

We can show that $Z_M$ is a structural p-consequence operation on the language $\mathcal{L}$. However, it is easy to show that when $D_1 \neq D_2$, the Tarski condition put on consequence operation: $Z_M(Z_M(X)) \subseteq Z_M(X)$, does not hold (cf. for example [1]). For example, consider the language $\mathcal{L} = (L, \rightarrow)$ with a binary connective “$\rightarrow$” and a p-matrix $\mathcal{M} = \{(0,1/3,2/3,1), \rightarrow, (2/3,1), \{1/3,2/3,1\}\}$, where for all $x, y \in \{0,1/3,2/3,1\}$: $x \rightarrow y = \min(1, 1-x+y)$. It is easy to check that the \textit{modus ponens} rule of inference is a valid rule of $Z_M$, in the sense that for any $\alpha, \beta \in L : \beta \in Z_M(\alpha, \alpha \rightarrow \beta)$. Then $s \in Z_M(Z_M(q \rightarrow r, q, q \rightarrow (r \rightarrow s))$ and $s \notin Z_M(q \rightarrow r, q, q \rightarrow (r \rightarrow s))$ (where $s, q, r \in \text{Var}, s \neq q$); put for example $hs = 0, hq = hr = 2/3$.

More generally, for any class $\mathcal{M} = \{\mathcal{M}_t : t \in T\}$ of p-matrices for the language $\mathcal{L} = (L, f_1, \ldots, f_n)$ let us define a structural p-consequence $Z_M : 2^L \rightarrow 2^L$ determined by $\mathcal{M}$ as follows: $\alpha \in Z_M(X)$ if and only if $\forall t \in T(\alpha \in Z_{\mathcal{M}_t}(X))$, that is $Z_M(X) = \bigcap\{Z_{\mathcal{M}_t}(X) : t \in T\}$.

**Definition 5.** For a given language $\mathcal{L} = (L, f_1, \ldots, f_n)$, a p-rule of inference $r$ is said to be valid for a p-consequence $Z_M$ (or simply, is a p-rule of $Z_M$), where $\mathcal{M} = \{\mathcal{M}_t : t \in T\}$ is a class of p-matrices of the form $\mathcal{M}_t = (M_t, F_{t,1}, \ldots, F_{t,n}, D_{t,1}^1, D_{t}^2)$ iff for each $t \in T$ and for each homomorphism $h_t$ from $\mathcal{L}$ into the algebra of $\mathcal{M}_t$ and for any n-tuple $(a_1, \ldots, a_n) \in r(n \geq 1)$: $h_t(pr_1(a_n)) \in D_{t,pr_2(a_n)}^1$ whenever for each $i \in \{1, 2, \ldots, n-1\}$, $h_t(pr_1(a_i)) \in D_{t,pr_2(a_i)}^1$ (in case $n = 1$, $h_t(pr_1(a_1)) \in D_{t,pr_2(a_1)}^1$). The set of all valid p-rules for a p-consequence $Z_M$ will be denoted by $R(M)$.

In that way one can obtain the following
Corollary. For any $p$-rule $r$ of a $p$-consequence $Z_M$, and for any sequence $(a_1,\ldots,a_n) \in r : pr_1(a_n) \in Z_M(pr_1(a_1),\ldots,pr_1(a_{n-1}))(n = 1 : pr_1(a_1) \in Z_M(\emptyset))$.

The following theorem corresponds to the well-known fact concerning that ordinary consequence operations namely that a derivability relation determined by the inferential base composed of all the rules of inference of a given consequence operation is weaker than that consequence operation.

**Theorem 2.** For any class of $p$-matrices $M = \{(M_t,F_{t,1},\ldots,F_{t,n},D_t^1,D_t^2) : t \in T\}$ and any $X \subseteq L$, $\alpha \in L$:

(i) $X \models_{\mathcal{R}(M)} \alpha \Rightarrow \alpha \in Z_M(X)$. (ii) $\alpha \in Z_M(X)$.

**Proof:** (i) Assume that $X \models_{\mathcal{R}(M)} \alpha$. Then there exists a $p$-proof $(a_1,\ldots,a_k)$ of $\alpha$ from the set $X$ on the basis of the set $\mathcal{R}(M)$ of all $p$-rules of $Z_M$ such that $pr_1(a_k) = \alpha$. To show that $pr_1(a_k) \in Z_M(X)$ choose any $t \in T$ and suppose that $h_t(X) \subseteq D_t^1$. It is sufficient to show (by induction) that for each $j = 1,\ldots,k$ : $(\ast) h_t(pr_1(a_j)) \in D_{pr_2(a_j)}^1$. If $j = 1$, either $pr_1(a_1) \in X$ and $pr_2(a_1) = 1$, or (b) for some $p$-rule $r \in \mathcal{R}(M)$, $\alpha \in r$. If (a) holds from the assumption it follows that $h_t(pr_1(a_1)) \in D_{pr_2(a_1)}^1$. If (b) holds then $(\ast)$ is valid since $r$ is a $p$-rule of $Z_M$.

Now it is enough to show the following condition:

for any $j \in \{2,\ldots,k\}$ : $h_t(pr_1(a_j)) \in D_{pr_2(a_j)}^1$ whenever $h_t(pr_1(a_1)) \in D_{pr_2(a_1)}^1,\ldots,h_t(pr_1(a_{j-1})) \in D_{pr_2(a_{j-1})}^1$.

Suppose that $h_t(pr_1(a_1)) \in D_{pr_2(a_1)}^1,\ldots,h_t(pr_1(a_{j-1})) \in D_{pr_2(a_{j-1})}^1$. Then either $pr_1(a_j) \in X$ and $pr_2(a_j) = 1$, or for some $p$-rule $r \in \mathcal{R}$ : $(a_{j_1},\ldots,a_{j_k}) \in r$ and $j_s = j,\{j_1,\ldots,j_{(k-1)}\} \subseteq \{1,\ldots,j-1\}$. Obviously, in both cases: $h_t(pr_1(a_j)) \in D_{pr_2(a_j)}^1$.

(ii) To prove that $Z_M$ is finitary and $\alpha \in Z_M(X)$. Then for some finite $X_f \subseteq X : \alpha \in Z_M(X_f)$. Let us put $X_f = \{\chi_1,\ldots,\chi_n\}$. Hence we have for all $t \in T : h_t\alpha \in D_1$ whenever $h_t\chi_1,\ldots,h_t\chi_n \in D_1$ for any homomorphism $h_t$ from $\mathcal{L}$ into $\mathcal{M}_t$. Thus there exists $r \in \mathcal{R}(M)$ of the form $\{<\chi_i,1><\alpha_1><\alpha,*>\}$, and the $p$-inference $<\chi_i,1><\alpha_1><\alpha_4*>$ is a $p$-proof of $\alpha$ from $X$ on the basis of $\mathcal{R}(M)$. □
In general case we have the following:

**THEOREM 3.** For every structural \( p \)-consequence \( Z \) there exists a class \( \Pi \) of \( p \)-matrices such that \( Z = Z_\Pi \).

**PROOF:** Consider a structural \( p \)-consequence \( Z \). Put \( \Pi := \{(L, Y, Z(Y)) : Y \subseteq L\} \). Everyone can show that \( Z = Z_\Pi \). \( \square \)

The representation of any structural \( p \)-consequence by a class of matrices as well as Theorems 1, 2 offers opportunity to ask about relation between two sets of \( p \)-rules: \( \mathcal{R}(Z_M) \) and \( \mathcal{R}(M) \). A simple example shows that \( \mathcal{R}(Z_M) \nsubseteq \mathcal{R}(M) \). Consider any 1-element class of \( p \)-matrices of the form: \( M = \{\{(0,1,2), F_1, \ldots, F_n, \{2\}, \{0,1,2\}\}) \). Then, for any \( X \subseteq L : Z_M(X) = L \). So for each formula \( \alpha \), the 1-element \( p \)-rule \( \{\langle \alpha, 1 \rangle \} \) is an element of \( \mathcal{R}(Z_M) \). However, \( \{(\langle p, 1 \rangle) \} \nsubseteq \mathcal{R}(M) \), where \( p \) is a sentential variable, since there exists a homomorphism \( h \) from the language into the algebra \( \{(0,1,2), F_1, \ldots, F_n\} \) such that \( hp \neq 2 \).

The following theorem says that the converse inclusion holds.

**THEOREM 4.** For any class of \( p \)-matrices \( M = \{(M_1, F_{t,1}, \ldots, F_{t,n}, D_1^t, D_2^t) : t \in T\} : \mathcal{R}(M) \subseteq \mathcal{R}(Z_M) \).

**PROOF:** Assume that \( r \in \mathcal{R}(M) \) and \((a_1, \ldots, a_n) \in r \). Suppose that for a set of formulas \( X \):

1. \( A_n(n - 1) \subseteq Z_M(X) \) and
2. \( Z_M(X \cup A_1(n - 1)) = Z_M(X) \).

Consequently it follows that

3. \( \forall t \in T \forall h_t \in \mathcal{H}(\mathcal{L}, M_t) [h_t(pr_1(a_1)) \in D_2^{pr_2(a_1)} \& \ldots \& h_t(pr_{\alpha-1}(a_{\alpha-1})) \in D_2^{pr_{\alpha-1}(a_{\alpha-1})}] : h_t(a_\alpha) \in D_1^t \).
4. \( \forall \alpha \in A_1(n - 1) : h_t(\alpha) \in D_1^t \). Furthermore,
5. \( h_t(X) \subseteq D_1^t \).

It implies that for each \( \alpha \in A_n(n - 1) : h_t(\alpha) \in D_1^t \), due to (1). In that way \( h_t(pr_1(a_1)) \in D_2^{pr_2(a_1)} \& \ldots \& h_t(pr_{\alpha-1}(a_{\alpha-1})) \in D_2^{pr_{\alpha-1}(a_{\alpha-1})} \) which yields \( h_t(pr_1(a_\alpha)) \in D_2^{pr_2(a_\alpha)} \) by (3) that is \( h_t(pr_1(a_\alpha)) \in D_1^t \). Hence it follows that \( h_t(\beta) \in D_1^t \).
(ii) Now let \( pr_2(a_n) = \ast \). Then one can prove that \( pr_1(a_n) \in Z_M(X) \) using (2) and under the assumption that \( h_t(X \cup A_1(n-1)) \subseteq D'_1 \), obtaining – along the lines (1), (5) and (3) that \( h_t(pr_1(a_n)) \in D'_{pr_2(a_n)}. \) \( \square \)

The main difference between the classes \( \mathcal{R}(Z) \) and \( \mathcal{R}(M) \) consists in the fact that we do not have any procedure to check whether \( r \in \mathcal{R}(Z) \) or \( r \not\in \mathcal{R}(Z) \), and \( \mathcal{R}(M) \) exists only for structural \( p \)-consequences.

3. A \( p \)-consequence related to \( n \)-valued Lukasiewicz logic

Now, let us provide an example of \( p \)-derivability relation and \( p \)-consequence. Consider sentential language \( \mathcal{L} = (L, \to, \lor, \land, \neg) \). Let \( \mathcal{R} \) be the set of following \( p \)-rules:

\[
\begin{align*}
A_1 &= \{< \alpha, 1 > : \alpha \in \text{Ax}_n\}, \\
A_2 &= \{< \alpha \to_1 \neg \alpha \land (\neg \alpha \to \alpha) \to \alpha, \ast > : \alpha \in L\},
\end{align*}
\]

where \( n > 2 \) is an odd natural number, \( \text{Ax}_n \) is the set of axioms adequate for \( n \)-valued Lukasiewicz logic \( L_n \) (cf. [6], \( n \)-valued Lukasiewicz logic is axiomatizable for all natural \( n \)),

\[
\begin{align*}
p(MP)_1 &= \{< \alpha \to \beta, 1 >, < \alpha, 1 >, < \beta, 1 > : \alpha, \beta \in L\}, \\
p(MP)_2 &= \{< \alpha \to \beta, 1 >, < \alpha, \ast >, < \beta, \ast > : \alpha, \beta \in L\}, \\
p(MP)_3 &= \{< \alpha \to \beta, \ast >, < \alpha, 1 >, < \beta, \ast > : \alpha, \beta \in L\}.
\end{align*}
\]

We show that \( \mathcal{R} \) is adequate with respect to following \( p \)-matrix:

\[
\mathcal{M}_n^\ast = \{\{0,1/n - 1, 2/n - 1, \ldots, n - 2/n - 1, 1\}, \to, \lor, \land, \neg, \{1\}, \{n - 1/2(n - 1), \ldots, n - 2/n - 1, 1\}\},
\]

where \( \mathcal{M}_n = (\mathcal{M}_n, \{1\}), \mathcal{M}_n = (\{0,1/n - 1, 2/n - 1, \ldots, n - 2/n - 1, 1\}, \to, \lor, \land, \neg) \), is \( n \)-valued Lukasiewicz matrix: \( x \to y = \text{min}(1,1 - x + y), x \lor y = \text{max}(x,y), x \land y = \text{min}(x,y), \neg x = 1 - x, x \to_k y = \text{min}(1,k(1 - x) + y) \).

**Soundness Theorem:** For all \( X \subseteq L, \alpha \in L : X \models_{\mathcal{R}} \alpha \Rightarrow \alpha \in Z_{\mathcal{M}_n^\ast}(X) \).

**Proof:** Since for any two sets of \( p \)-rules \( \mathcal{R}_1, \mathcal{R}_2 \) for a given language: \( \models_{\mathcal{R}_1} \subseteq \models_{\mathcal{R}_2} \) whenever \( \mathcal{R}_1 \subseteq \mathcal{R}_2 \), so accordingly Theorem 2, in order to prove soundness theorem it is sufficient to show that every \( p \)-rule from \( \mathcal{R} \) is a rule of the \( p \)-consequence \( Z_{\mathcal{M}_n^\ast} \).
It is obvious that for any $a \in A_1: \forall h \in Hom(\mathcal{L}, M_n), h(pr_1(a)) = 1$, which implies that $A_1$ is a p-rule of $Z_{\mathcal{M}_2}$. Now suppose that for some homomorphism $h$ and a formula $\alpha$: $[(h\alpha \rightarrow n_{-2} \neg h\alpha) \land (\neg h\alpha \rightarrow h\alpha)] \rightarrow h\alpha < 1/2$. Then $1/2 + h\alpha < \min\{\min[1, (n-2)(1-h\alpha)+1-h\alpha], \min[1, 1-(1-h\alpha)+h\alpha]\}$. This yields $1/2 + h\alpha < 1$, that is $h\alpha < 1/2$. However $1/2 + h\alpha < \min\{\min[1, (n-1)(1-h\alpha)], \min[1, 2h\alpha]\}$. Since when $h\alpha < 1/2$ it follows that $(n-1)(1-h\alpha) > 1$ and $2h\alpha < 1$, so $1/2 + h\alpha < \min[1, 2h\alpha] = 2h\alpha$ but this implies that $1/2 < h\alpha$: a contradiction. In that way, for any $a \in A_2: \forall h \in Hom(\mathcal{L}, M_n), h(pr_1(a)) \in \{n-1/2(n-1), \ldots, n-2/n-1, 1\}$ therefore $A_2$ is a p-rule of $Z_{\mathcal{M}_2}$. In order to show that for $i = 1, 2, 3, p(MP)_i$ is a p-rule of $Z_{\mathcal{M}_2}$, one can verify for any $h \in Hom(\mathcal{L}, M_n)$ the following 3 cases:

(i) $h\alpha = h\alpha \rightarrow h\gamma = 1$, then obviously $h\beta = 1$

(ii) $h\alpha \rightarrow h\beta = 1$ and $h\alpha \geq 1/2$, then $1 - h\alpha + h\beta \geq 1$, so $h\beta > h\alpha \geq 1/2$

(iii) $h\alpha \rightarrow h\beta \geq 1/2$ and $h\alpha = 1$, then $1 - h\alpha + h\beta \geq 1/2$ and consequently $h\beta \geq 1/2$. □

**Completeness Theorem:** For all $X \subseteq L, \alpha \in L: \alpha \in Z_{\mathcal{M}_2}(X) \Rightarrow X \models_\mathcal{R} \alpha$.

**Proof:** Let $a \in Z_{\mathcal{M}_2}(X)$. Then $\forall h(Hom(\mathcal{L}, M_n))h(X) \subseteq \{1\} \Rightarrow h\alpha \in \{n-1/2(n-1), \ldots, n-2/n-1, 1\}$. Let us put $a_0 := [(\alpha \rightarrow n_{-2} \neg \alpha) \land (\neg \alpha \rightarrow \alpha)] \lor \alpha$. Then one can obtain that $\forall h \in Hom(\mathcal{L}, M_n)|h(X) \subseteq \{1\} \Rightarrow h\alpha = 1$. Namely, when $h(X) \subseteq \{1\}$, that is $h\alpha \geq 1/2$, the following holds true:

(1) $h((\alpha \rightarrow n_{-2} \neg \alpha) \land (\neg \alpha \rightarrow \alpha)) = \min\{\min[1, (n-1)(1-h\alpha)], \min[1, 2h\alpha]\} = 1$ if $h\alpha \neq 1$, and $h((\alpha \rightarrow n_{-2} \neg \alpha) \land (\neg \alpha \rightarrow \alpha)) = 0$ if $h\alpha = 1$, which yields $h(((\alpha \rightarrow n_{-2} \neg \alpha) \land (\neg \alpha \rightarrow \alpha)) \lor \alpha) = 1$. Thus $a_0 \in L_n(X)$. So from the completeness theorem for n-valued Lukasiewicz logic it follows that there exists a proof $(\alpha_1, \ldots, \alpha_r, a_0)$ of $a_0$ from the set $X$ on the basis of the rules $Ax_n$ and modus ponens. So the sequence $(< \alpha_1, 1 >, < \alpha_0, 1 >)$ is a p-proof of $a_0$ from the set $X$ on the basis of the set of p-rules $\{Ax_n, p(MP)_1\}$, i.e., on the basis of the set $\mathcal{R}$.

Now let us consider a formula $\Xi := \beta_0 \rightarrow (a_0 \rightarrow \alpha)$, where $\beta_0 := [(\alpha \rightarrow n_{-2} \neg \alpha) \land (\neg \alpha \rightarrow \alpha)] \rightarrow \alpha$. Then $\Xi \in L_n(X)$. Indeed, suppose that for some $h: \mathcal{L} \rightarrow M_n : h(X) \subseteq \{1\}$ and $h(\Xi) \neq 1$. Then from the earlier results it follows that $h\alpha \geq 1/2, h\alpha_0 = 1$. Moreover, from the definition of $\Xi : h\beta_0 \rightarrow (h\alpha_0 \rightarrow h\alpha) < 1$. Hence, $h\alpha_0 \rightarrow h\alpha < h\beta_0$ and consequently, $1 \rightarrow h\alpha < h\beta_0$, which means that $h\alpha < h\beta_0$. Thus $h\alpha < 1$. However, then
according to (1) and the definition of $\beta_0: h\beta_0 = h((\alpha \rightarrow \neg \alpha) \wedge (\neg \alpha \rightarrow \alpha)) \rightarrow h\alpha = 1 \rightarrow h\alpha$. In that way, $h\alpha < 1 \rightarrow h\alpha$, and since $h\alpha < 1$ so $1 - 1 + h\alpha < 1$ which means that $1 \rightarrow h\alpha = h\alpha$ and consequently, $h\alpha < h\alpha$.

Finally, there exists a $p$-proof $(< \beta_i, 1 >_{i=1}^s, < \Xi, 1 >)$ of the formula $\Xi$ from $X$ on the basis of $R$, where $(\beta_1, \ldots, \beta_s, \Xi)$ is an ordinary proof of $\Xi$ from $X$ on the basis of $Ax_n$ and modus ponens. Then the sequence:

$$(< \beta_1, 1 >, \ldots, < \beta_s, 1 >, < \Xi, 1 >, < \beta_0, \ast > (p(MP)_2), < \alpha_1, 1 >, \ldots, < \alpha_r, 1 >, < \alpha_0, 1 >, < \ast > (p(MP)_3))$$

is a $p$-proof of $\alpha$ from $X$ on the basis of the set of $p$-rules $R$, which yields $X \parallel_R \alpha$.

In particular, for $n = 3$ the set $R$ of the following $p$-rules (cf. [3, p. 179]) is adequate for the $p$-consequence determined by the $p$-matrix $M^i_3 = ([0, 1/2, 1], \to, \lor, \land, \neg, [1], \{1/2, 1\})$:

$$A_1 = \text{the union of the sets:}$$

$$\{ < \alpha \rightarrow (\beta \rightarrow \alpha), 1 > : \alpha, \beta \in L \},$$

$$\{ < (\alpha \rightarrow \beta) \rightarrow ((\beta \to \gamma) \rightarrow (\alpha \rightarrow \gamma)), 1 > : \alpha, \beta, \gamma \in L \},$$

$$\{ < (\neg \alpha \rightarrow \neg \beta) \rightarrow (\beta \rightarrow \alpha), 1 > : \alpha, \beta \in L \},$$

$$\{ < ((\alpha \rightarrow \neg \alpha) \rightarrow \alpha), 1 > : \alpha, \beta \in L \},$$

$$A_2 = \{ < (\alpha \leftrightarrow \neg \alpha) \rightarrow \alpha, \ast > : \alpha, \beta \in L \},$$

$$p(MP)_1 \cup p(MP)_2 \cup p(MP)_3.$$
Formalization of a Plausible Inference

\[ n(x) = \begin{cases} 
  m - 1, & \text{if } x = 0 \\
  x - 1, & \text{if } x \neq 0, 
\end{cases} \quad a(x, y) = \max(x, y). \]

By an \( m \)-element Post’s \( p \)-matrix for the sentential language \( L = (L, \neg, \lor) \), we shall understand any structure of the form: \( M^m_{i, r} = \langle \{0, 1, \ldots, m - 1\}, n, a, \rangle \{m - i, m - i + 1, \ldots, m - 1\}, \{m - r, m - r + 1, \ldots, m - i, m - i + 1, \ldots, m - 1\} \rangle \), where \( 1 \leq i < r < m \) and \( M^m_{i, r} = \langle \{0, 1, \ldots, m - 1\}, n, a, \rangle \{m - i, m - i + 1, \ldots, m - 1\} \rangle \) is the \( m \)-element Post’s matrix.

**Theorem 5.** For any set of the ordinary rules of inference \( \mathcal{R} \) adequate for the consequence operation determined by the matrix \( M^m_{i, r} \) and for any \( r \in \{i + 1, \ldots, m - 1\} \), there exists a set of \( p \)-rules \( \Phi_r(\mathcal{R}) \) such that for all \( X \subseteq L, \alpha \in L : X \models^p_{\Phi_r(\mathcal{R})} \alpha \) iff \( \alpha \in Z(X) \), where \( Z \) is a \( p \)-consequence determined by the \( p \)-matrix \( M^m_{i, r} \).

**Proof:** Assume that for the consequence \( C \) determined by the matrix \( M^m_{i, r} \), a set of the ordinary rules of inference \( \mathcal{R} \) is such that

1. \( X \models^p \alpha \) iff \( \alpha \in C(X) \), for all \( X \subseteq L, \alpha \in L \), where \( \models^p \) is the ordinary provability relation determined by \( \mathcal{R} \). Let us put \( \Phi_r(\mathcal{R}) := \{< \alpha_j, 1 >_{j=1}^n \} : < \{\alpha_1, \ldots, \alpha_{n-1}\}, \alpha_n \rangle \in \rho \) for some \( \rho \in \mathcal{R} \} \cup \{< \alpha \lor \bigvee_{j=m+1}^{m+r+1} \neg_j \alpha, 1 >, < \alpha, \ast > : \alpha \in L \} \rangle \).

\[ \Rightarrow \] (soundness): According to Theorem 2 it is sufficient to show that every \( p \)-rule from \( \Phi_r(\mathcal{R}) \) is a \( p \)-rule of the \( p \)-consequence \( Z \). The only case worth checking is that

2. \( h\alpha \in \{m - r, m - r + 1, \ldots, m - i, m - i + 1, \ldots, m - 1\} \) when \( h(\alpha \lor \bigvee_{j=m+1}^{m+r+1} \neg_j \alpha) \in \{m - i, m - i + 1, \ldots, m - 1\} \).

for all homomorphisms \( h \) from \( (L, \neg, \lor) \) into \( \{\{0, 1, \ldots, m - 1\}, n, a\} \). To this aim first consider the following tableau providing the values for some superpositions of the operation \( n \):

<table>
<thead>
<tr>
<th>( x = )</th>
<th>0</th>
<th>1</th>
<th>\ldots</th>
<th>( m - r - 2 )</th>
<th>( m - r - 1 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( n^{m-1}(x) )</td>
<td>1</td>
<td>2</td>
<td>\ldots</td>
<td>( m - r - 2 )</td>
<td>( m - r - 1 )</td>
</tr>
<tr>
<td>( n^{m-2}(x) )</td>
<td>2</td>
<td>3</td>
<td>\ldots</td>
<td>( m - r )</td>
<td>( m - r + 1 )</td>
</tr>
<tr>
<td>\ldots</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( n^{m-r}(x) )</td>
<td>( r - i )</td>
<td>( r - i + 1 )</td>
<td>\ldots</td>
<td>( m - i - 2 )</td>
<td>( m - i - 1 )</td>
</tr>
</tbody>
</table>
On the basis of the tableau one can establish that for any homomorphism $h$ from $(L, \neg, \lor)$ into $(\{0, 1, \ldots, m - 1\}, n, a)$ and any formula $\alpha$ the following condition holds:

\[
(3) \quad h\alpha \in \{m-r, \ldots, m-1\} \text{ iff } h(\alpha \lor \bigvee_{j=m-1}^{m-r+i} \neg_j \alpha) \in \{m-i, \ldots, m-1\}.
\]

Clearly, (2) follows directly from (3).

$(\Leftarrow)$ (completeness): Assume that $\alpha \in Z(X)$. Consider any homomorphism $h$ such that $h(X) \subseteq \{m-i, \ldots, m-1\}$. Then $h\alpha \in \{m-r, \ldots, m-1\}$. So from (3): $h(\alpha \lor \bigvee_{j=m-1}^{m-r+i} \neg_j \alpha) \in \{m-i, \ldots, m-1\}$. Thus \(\alpha \lor \bigvee_{j=m-1}^{m-r+i} \neg_j \alpha \in C(X)\). In that way on the basis of (1) and the definition of the set $\Phi_r(\mathcal{R})$ it follows that there exists a $p$-proof $(a_1, \ldots, a_k)$ of $\alpha \lor \bigvee_{j=m-1}^{m-r+i} \neg_j \alpha$ from $X$ on the basis of $\Phi_r(\mathcal{R})$ such that $pr_2(\{a_1, \ldots, a_k\}) = 1$. So the sequence $(a_1, \ldots, a_k, a_{k+1})$, where $pr_1(a_{k+1}) = \alpha, pr_2(a_{k+1}) = \ast$ is a $p$-proof of the formula $\alpha$ from the set $X$ on the basis of the set of $p$-rules $\Phi_r(\mathcal{R})$. \qed

References


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