A MODAL EXTENSION OF FIRST ORDER CLASSICAL LOGIC–Part I

Abstract
We formalize a fragment of the metatheory of classical first order logic by adding a new connective, the modal □, whose intended interpretation is the (classical) metalogical predicate “is a theorem” (⊬). We then illustrate how to employ this modal extension of classical logic to write equational proofs of classical theorems, and prove (in Part II) its completeness and soundness with respect to Kripke models. We also prove that the extension meets our objective: For any classical formulae \( A \) and \( B \), we can prove \( □A \rightarrow □B \) modally iff we can prove \( A \vdash B \) classically.

Keywords: First order logic, modal logic, equational logic, calculational logic, consistency, completeness, Leibniz rule, derivability conditions, provability predicate, Kripke models.

1. Introduction

First order equational, or calculational, logic was proposed in [4], [6], [7] and was shown to be sound and complete in [13], [14], [15]. As a practitioner’s tool, it relies heavily on Leibniz’s principle of “replacing equals by equals” allowing the user to prove assertions in the same manner that one verifies the equality or inequality of two expressions in high school algebra.

1This research was partially supported by NSERC grant No. 8820
or trigonometry, namely, by constructing a conjunctional\(^2\) chain of equalities and inequalities—in the case of logic, equivalences (\(\leftrightarrow\)) and implications (\(\rightarrow\)) respectively. A classical formal proof in the equational style ought to look like

\[
A_1 \circ A_2 \circ \ldots \circ A_n
\]

normally written vertically, where each occurrence of “\(\circ\)” stands for one of the formal connectives “\(\rightarrow\)” or “\(\leftrightarrow\)”. Such a configuration establishes the derivability of \(A_n\) if \(A_1\) is known to be a theorem. More generally, it establishes the derivability of \(A_1 \rightarrow A_n\) (\(A_1 \leftrightarrow A_n\) if all the \(\circ\) are \(\leftrightarrow\)).

In practice, one often needs to break this chain of formal equivalences and implications and invoke a rule of inference that will spawn a new chain, disjoint from the original. For example,

\[
A_1 \circ \ldots \circ B \bullet (\forall x)B \circ \ldots \circ A_n
\]

The “\(\bullet\)” above cannot be either of \(\rightarrow\) or \(\leftrightarrow\), but is the classical metalogical \(\vdash\) (here also the two-sided \(\vdash \dashv\) works). Thus the configuration (2) proves \(A_n\) from \(A_1\), however it does so using two “connected components” of type (1). It is “disconnected” at the point where one moves into the metatheory and applies \(\vdash \dashv\).

We believe that from a methodological point of view it is desirable to have a new proof calculus that is sufficiently powerful to simulate classical equational proofs such as (2) above without having to disconnect the proof by the insertion of the metalogical connectives \(\vdash\) or \(\vdash \dashv\), as the latter are often confused with the formal \(\rightarrow\) and \(\leftrightarrow\) by inexperienced users. We expect that such a calculus will eliminate this potential confusion by simplifying and further mechanizing the process of writing equational proofs, as the user will now exclusively deal with the formal \(\rightarrow\) and \(\leftrightarrow\). Similar comments were contained in [8] were it was, essentially, suggested that one formalize \(\vdash\) as the modal \(\Box\), extending in some natural way the classical logic (such an extension was restricted to the propositional case in loc. cit.).

Our goal in this paper is to present a simple modal framework within which we can write equational proofs of classical theorems\(^3\) in a manner

\(^2\)E.g., \(a = b < c = d < e\), meaning \(a = b\) and \(b < c\) and \(c = d\) and \(d < e\).

\(^3\)The arbitrary modal equational proof may also be disconnected. However we are only interested in modal proofs that simulate classical equational proofs. Such simulations will be always connected.
that the only connectives used are the formal \( \rightarrow \) and \( \leftrightarrow \). In this framework the two formal chains in (2) that were disconnected at “•” will be rendered by the connected modal chain

\[
\Box A_1 \text{ annotation } \ldots \text{ annotation } \Box B \leftrightarrow \Box (\forall x) B \text{ annotation } \ldots \text{ annotation } \Box A_n
\]

(3)

To this end we devise a particular modal extension of classical first order logic that simulates generalization in a natural manner and prove that it meets our main conservation requirement. For any classical formulae \( A \) and \( B \) and classical theory \( T \), we have that \( T \cup \{ A \} \) proves \( B \) classically iff \( T \) proves \( \Box A \rightarrow \Box B \) modally.

Fulfillment of this requirement allows one to replace any, possibly disconnected, classical equational proof by a connected modal equational proof: First, replace all classical \( \rightarrow \) and \( \leftrightarrow \) by the classical metalogical \( \vdash \) and \( \vdash \not\vdash \) respectively (a valid step by modus ponens). Then replace every \( A \vdash B \) (respectively, \( A \vdash \not\vdash B \)) by \( \Box A \rightarrow \Box B \) (respectively, \( \Box A \leftrightarrow \Box B \)).

It is immediate by the conservation requirement that, conversely, a connected modal chain of type (3) can be replaced by a classical, possibly disconnected, chain of type (2).

Modal extensions of propositional ([2], [8], [9], [10], [12], [17]) and predicate ([1], [3], [5], [9], [10], [11]) classical logic are not new, however, previous ones differ in at least two significant ways from our approach. One, they do not address, but we do, the needs of the “user”, investigating instead general or specific philosophical and metamathematical issues. Two, their “\( \Box \)” either denotes (intuitively) a form of “necessity”, or it denotes provability in the narrow sense of Gödel’s provability predicate for Peano arithmetic (or for ZF). As far as we know, our approach is the first that uses “\( \Box \)” to simulate classical first order provability in general. This entails some departures from the norm: Firstly, the semantics of Section 5 (in Part II) leads to models that are similar to but not identical to varying domain models. Secondly, we require \( \Box A \) to be a sentence for all formulae \( A \) (contrast, e.g., with [1], [3], [5]), a choice that we justify later on.

There are many variants of modal logic to choose from as a starting point. We build upon a first order version of logic K4, adding just one axiom schema ((M3) of Section 2) that simulates classical generalization \( A \vdash (\forall x) A \). The conservation requirement determines the form of (M3):
\( \Box A \rightarrow \Box(\forall x)A \).\(^4\) This axiom plays a role in making Gödel completeness work (cf. Lemma 6.3 in Part II).

Motivation for keeping the two modal axioms of K4 is straightforward: Axiom (M1) in Section 3 simulates classical modus ponens (cf. 4.10). Axiom (M2), of less obvious intuitive value, is technically expedient, for example, towards the proof of weak necessitation and inner Leibniz rule. These two axioms are the counterparts of Löb’s derivability conditions DC2 and DC3 respectively that one encounters in proofs of Gödel’s second incompleteness theorem ([2], [12], [16]), this observation providing immediate peace of mind with respect to their relative consistency with the classical axioms. By the way, our axiom (M3) too is “true” when \( \Box \) is interpreted as Gödel’s provability predicate.\(^5\) This observation, once more, puts to rest any consistency worry.

We do not need the reflection principle—that is, axiom schema \( \Box A \rightarrow A \) of logic S4—and we do not include it. Indeed, it is a trivial exercise to show that it is underivable in our logic, for one can easily build a countermodel (using an irreflexive frame relation) by the techniques of Section 5 of Part II. We also prefer to simulate the inference “if \( A \), then \( \Box A \)” (corresponding to Löb’s DC1) by hiding it inside the axioms, adopting only classical primitive rules of inference.

The paper is split into two parts. Part I (Sections 2–4) deals with syntactic issues. Part II (Sections 5–6) introduces semantics, proves the soundness and strong completeness of the proposed logic with respect to Kripke structures, and proves that the main conservation requirement is met.

2. The Language of Modal Logic

Terms are built from the object variables \( x, y, z, x', y'', z_3, \ldots \), and whatever nonlogical symbols may be available in any particular theory of interest, such as constants \( a, b, c, a'', c_1, \ldots \), and functions \( f, g, h, f', h''_{10}, \ldots \), exactly as in classical first order logic.

\(^4\)Since \( \Box A \) is closed for all \( A \), one may view (M3) as a special case of the Barcan formula.

\(^5\)“True” abuses language here, and it means: If \( P(x) \) is Gödel’s provability predicate for Peano arithmetic, and if \( A \) denotes the (formal) Gödel number of the formula \( A \), then one can prove \( P( A ) \rightarrow P( \forall x)A \) in some appropriate conservative extension of Peano arithmetic ([16]).
Formulae (well-formed modal formulae–wfmf) are built, via the standard inductive definitions, from atomic formulae–which may involve predicate symbols $P, Q, R, P''_3, \ldots$–and the primary logical symbols. The latter are the Boolean variables $p, q, p', p'', q_{13}, \ldots$, and the connectives: $\neg, \lor, \top, \bot, \Box, (,) =, \forall$, and the comma.

We note two slight deviations from the standard definitions: One is that we add an induction clause “if $A$ is formula, then so is $(\Box A)$”. The other is the inclusion of Boolean variables (traditionally used in equational logics to facilitate applications of the Leibniz rule) and the Boolean constants $\top$ (formal “true”) and $\bot$ (formal “false”). Every Boolean symbol is an atomic formula. $A$ is the scope of the leftmost $\Box$ in $(\Box A)$. Every object variable in the scope of a $\Box$ is bound; thus, $(\Box A)$ is closed (a sentence).

The motivation regarding object variables is our intended intuitive interpretation of $\Box$ as the classical $\vdash$, and therefore as the classical $|=\,$ as well. When we say “$|= A$” classically, we mean that for all structures where we interpret $A$, and for all value-assignments to the free object variables of $A$, the formula is true. Thus the variables in a statement such as “$|= A$” are implicitly universally quantified and are unavailable for substitutions. However, Boolean variables in a wfmf are always free.

If a formula does not contain $\Box$, then we say it is a classical formula–or a well-formed formula, or wff. Additional symbols $\land, \to, \leftrightarrow, \exists$ are introduced metalinguistically in the usual manner. The usual metalinguistic rules of how to omit brackets apply: highest priority symbols are the “unary” $\forall, \exists, \neg, \Box$; all associativities are right.

We denote substitutions into variables of a wfmf $A$ by the meta expressions $A[x := t]$ (term substituted into all occurrences of the free object variable $x$) and $A[p := B]$ (formula substituted into all occurrences of the Boolean variable $p$). In the the first type we disallow substitution if proceeding would have some variable of the term $t$ captured by a quantifier. In the second type we proceed regardless. In our notation the symbols “$[x := t]$” and “$[p := B]$” have the least scope, thus, e.g., $(\forall x)A[y := t]$–if allowed–means $(\forall x)(A[y := t])$.

If $\Delta$ is a set of wfmf, then $\Box \Delta$ denotes $\{\Box A : A \in \Delta\}$. We say that $\Box A$ is the “boxed version of $A$” and $\Box \Delta$ is the “boxed version of $\Delta$".
3. Axioms

We shall call our first order modal logic $M^3$, the “3” indicating the presence of three modal axioms.

**Definition 3.1.** The set of axioms of $M^3$ is $\Lambda \cup \Box \Lambda$, where $\Lambda$ consists of all instances of the following schemata.

1. All tautologies
2. $(\forall x)A \rightarrow A[x := t]$, provided the substitution is allowed
3. $A \rightarrow (\forall x)A$, provided $x$ is not free in $A$
4. $(\forall x)(A \rightarrow B) \rightarrow (\forall x)A \rightarrow (\forall x)B$
5. $x = x$
6. $s = t \rightarrow (A[x := s] \leftrightarrow A[x := t])$, provided the substitutions are allowed

(M1) $\Box(A \rightarrow B) \rightarrow \Box A \rightarrow \Box B$
(M2) $\Box A \rightarrow \Box \Box A$
(M3) $\Box A \rightarrow \Box (\forall x)A$

There are two primary rules of inference. Modus ponens (MP) “if $A$ and $A \rightarrow B$, then infer $B$”, and generalization (Gen) “if $A$, then infer $(\forall x)A$, for any object variable $x$”.

We shall always work within a mathematical theory, generically denoting its set of nonlogical axioms by $T$. Examples of $T$ are ZFC, Peano arithmetic, or something totally wild (including wfnf’s), or $\emptyset$. In the latter case we have a pure theory, i.e., we are doing just logic.

**Definition 3.2.** [$\Gamma$-proofs and $\Gamma$-theorems] We say that a formula $A$ is a $\Gamma$-theorem of $T$ based on a (possibly empty) set of additional assumptions, $\Gamma$– and write $\Gamma \vdash T A$–iff there is a $\Gamma$-proof of $A_n$–from $T$. By such a proof we understand a sequence of formulae $A_1, \ldots, A_n$ such that $A$ is $A_n$ and each $A_i$ satisfies one of the following conditions:

1. $A_i \in \Lambda \cup \Box \Lambda \cup T \cup \Box T \cup \Gamma$
2. There are numbers $j, k < i$ such that $A_k$ is $A_j \rightarrow A_i$
3. There is a number $j < i$ such that $A_i$ is $(\forall x)A_j$.

The corresponding direct recursive definition of $\Gamma$-theorems is to say that $A$ is a $\Gamma$-theorem iff it satisfies (1) (using “$A_i$” for “$A_i$”) or there is a $\Gamma$-theorem $B$, such that $B \rightarrow A$ is also a $\Gamma$-theorem, or $A$ is $(\forall x)B$ and $B$ is a $\Gamma$-theorem. We omit writing $\Gamma$ (or $T$) if it is empty.
Having added the boxed versions of all the axioms in $\mathcal{T}$ and $\Lambda$, it is unnecessary to include “if $A$, then $\Box A$” as a primary rule, since we can obtain a form of this inference as a derived rule (4.2 below). We also note that there is a subtle but important difference between writing $\Gamma \vdash A$ and $\models \Gamma A$, namely, $\models \Gamma A$ is the same as $\Gamma \cup \Box \Gamma \vdash A$.

4. Some metatheorems

Metatheorem 4.1. [Tautology Theorem] If $A_1, \ldots, A_n \models \text{taut } B$, then $A_1, \ldots, A_n \vdash_\mathcal{T} B$ for any $\mathcal{T}$.

Metatheorem 4.2. [Derived Rule: Weak Necessitation (WN)] If $\Gamma \vdash_\mathcal{T} A$, then $\Gamma \vdash \Box A$, provided $\Gamma = \Box \Gamma'$ or $\Gamma = \Gamma' \cup \Box \Gamma''$ for some $\Gamma'' \supseteq \Gamma'$.

Proof. Induction on $\Gamma$-theorems.

(1) If $A \in \Lambda \cup \mathcal{T}$, then $\Box A \in \Box (\Lambda \cup \mathcal{T})$, and we are done. If $A$ is $\Box B$ for some $B \in \Lambda \cup \mathcal{T}$, then we have $\vdash_\mathcal{T} \Box B$, but also $\vdash_\mathcal{T} \Box B \rightarrow \Box \Box B$, by (M2). Thus $\vdash_\mathcal{T} \Box \Box B$ by MP.

(2) If $A \in \Gamma$, then we proceed as in (1).

(3) Let $\Gamma \vdash_\mathcal{T} A$, and also $\Gamma \vdash_\mathcal{T} B$ and $\Gamma \vdash_\mathcal{T} B \rightarrow A$. We have $\Gamma \vdash_\mathcal{T} \Box B$ and $\Gamma \vdash_\mathcal{T} \Box (B \rightarrow A)$ by induction hypothesis (I.H.). Then we have $\Gamma \vdash_\mathcal{T} \Box B \rightarrow \Box A$ by (M1) and MP. Using MP again, we get $\Gamma \vdash_\mathcal{T} \Box A$.

(4) Let $\Gamma \vdash_\mathcal{T} C$, and $A$ be $(\forall x)C$. By I.H., $\Gamma \vdash_\mathcal{T} \Box C$, hence $\Gamma \vdash_\mathcal{T} \Box (\forall x)C$ by (M3) and MP.

Corollary 4.3. If $\vdash_\mathcal{T} A$, then $\vdash_\mathcal{T} \Box A$.

We call an inference rule weak if in order to obtain its conclusion we must place restrictions on how the premises were derived. Otherwise the rule is “strong”. For example, Gen is strong for we place no conditions on the hypothesis $A$.

Metatheorem 4.4. [Outer Deduction Theorem] For any formulae $A, B$ and any set of formulae $\Gamma$, if $\Gamma + A \vdash_\mathcal{T} B$ with a condition, then $\Gamma \vdash_\mathcal{T} A \rightarrow B$. The condition is that a $\Gamma + A$-proof of $B$ exists that contains no generalization steps $C \vdash (\forall x)C$, for any $x$ that is free in $A$.

\footnote{If $A_1, \ldots, A_n \models \text{taut } B$ indicates that $A_1, \ldots, A_n$ tautologically imply $B$. That is the same as saying that $\models \text{taut } A_1 \rightarrow \ldots \rightarrow A_n \rightarrow B$, i.e., that $A_1 \rightarrow \ldots \rightarrow A_n \rightarrow B$ is a tautology.}
N.B. \( \Gamma + A \) is an abbreviation of \( \Gamma \cup \{ A \} \).

**Proof.** By induction on \( \Gamma + A \)-theorems \( B \) obtained via \( \Gamma + A \)-proofs that satisfy the condition.

(1) If \( B \) is in one of \( \Lambda, \square \Lambda, T, \) or \( \square T \), then \( \vdash_T B \). Now, \( B \vdash_{\text{taut}} A \rightarrow B \). So we get \( \vdash_T A \rightarrow B \) by 4.1, from which \( \Gamma \vdash_T A \rightarrow B \).

(2) Suppose \( B \) is in \( \Gamma \). Then \( \Gamma \vdash_T B \). Since \( B \vdash_{\text{taut}} A \rightarrow B \), as above, we have \( \Gamma \vdash_T A \rightarrow B \).

(3) Suppose \( B \) is \( A \). Then \( A \rightarrow B \) is the tautology \( A \rightarrow A \). Hence \( \vdash_T A \rightarrow B \), and so \( \Gamma \vdash_T A \rightarrow B \).

(4) Suppose \( \Gamma + A \vdash_T C \) and \( \Gamma + A \vdash_T C \rightarrow B \). By I.H., \( \Gamma \vdash_T A \rightarrow C \) and \( \Gamma \vdash_T A \rightarrow (C \rightarrow B) \). Since \( A \rightarrow C, A \rightarrow (C \rightarrow B) \vdash_{\text{taut}} A \rightarrow B \), we have \( \Gamma \vdash_T A \rightarrow B \).

(5) Finally, let \( \Gamma + A \vdash D \) and \( (\forall x)D \) be \( B \). By I.H., \( \Gamma \vdash A \rightarrow D \), hence \( \Gamma \vdash (\forall x)(A \rightarrow D) \) by Gen. Axiom (4) now yields

\[
\Gamma \vdash (\forall x)A \rightarrow (\forall x)D
\]

via MP. The fact that \( D \vdash (\forall x)D \) was employed in the proof of \( B \) means that \( x \) is not free in \( A \). Thus, by axiom (3) and 4.1, (i) yields \( \Gamma \vdash A \rightarrow (\forall x)D \).\[\blacksquare\]

**Metatheorem 4.5.** [Inner Generalization] \( \vdash \square A \leftrightarrow \square (\forall x)A \).

**Proof.** The \( \leftarrow \) direction is by \( \square ((\forall x)A \rightarrow A) \) (boxed axiom (2)), by (M1) and MP. The \( \rightarrow \) direction is by (M3).\[\blacksquare\]

**Remark 4.6.** The qualifiers “outer” and “inner” are used with respect to the classical logic that \( M^3 \) extends. Thus, an inner “rule” is not a rule of \( M^3 \) at all. Rather it is a theorem schema of \( M^3 \) that simulates a rule of classical logic according to the (soon to be proved) main conservation requirement. Here “inner generalization” simulates classical generalization on classical wff \( A \): “\( A \) and \( (\forall x)A \) are mutually derivable”. Note however that 4.5 applies to all wfmf \( A \) not only to classical wff \( A \).

The qualifier “outer” applied to a derived rule indicates that, unlike “inner”, this is a bona fide derived rule of \( M^3 \).\[\blacksquare\]

**Metatheorem 4.7.** \( \vdash_T (\forall x)(A \leftrightarrow B) \rightarrow ((\forall x)A \leftrightarrow (\forall x)B) \).
Metatheorem 4.8. [□-monotonicity] If $\vdash \Box (A \to B)$, then $\vdash \Box A \to \Box B$.

Metatheorem 4.9. $[\Box \text{ over } \leftrightarrow] \vdash \Box (A \leftrightarrow B) \to (\Box A \leftrightarrow \Box B)$.

Proof.

\[
\begin{align*}
\Box (A \leftrightarrow B) \\
\to \langle 4.8 \text{ plus } \equiv \text{taut } (A \leftrightarrow B) \to A \to B \rangle \\
\Box (A \to B) \\
\to \langle (M1) \rangle \\
\Box A \to \Box B
\end{align*}
\]

We similarly prove $\vdash \Box (A \leftrightarrow B) \to (\Box B \to \Box A)$ and are done by 4.1.

Remark 4.10. $\vdash \Box (A \leftrightarrow B) \to (\Box A \to \Box B)$ is the counterpart of the equanimity rule of [6], [13], [14], [15], namely “$A \leftrightarrow B, A \vdash B$”. This follows from Boolean manipulation, the modal theorem $\Box (A \land B) \leftrightarrow \Box A \land \Box B$ (exercise) and the main conservation requirement. Similarly, (M1) is inner MP, capturing the classical “$A \to B, A \vdash B$”.

Metatheorem 4.11. [Outer $\forall$-monotonicity] If $\Gamma \vdash A \to B$, then $\Gamma \vdash (\forall x) A \to (\forall x) B$.

Metatheorem 4.12. [Inner $\forall$-monotonicity]

\[
\vdash \Box (A \to B) \to \Box ((\forall x) A \to (\forall x) B)
\]

Proof.

\[
\begin{align*}
\Box (A \to B) \\
\to \langle (M3) \rangle \\
\Box ((\forall x) (A \to B)) \\
\to \langle \text{boxed axiom (4), and } \Box \text{-monotonicity (4.8)} \rangle \\
\Box ((\forall x) A \to (\forall x) B)
\end{align*}
\]

Metatheorem 4.13. [Inner Leibniz Rule]

\[
\vdash \Box (A \leftrightarrow B) \to \Box (C[p := A] \leftrightarrow C[p := B])
\]
Proof. We prove the claim by induction on the formula \( C \).

Basis: If \( C \) is one of \( q \) (other than \( p \)), \( p \), \( \top \), \( \bot \), then the result follows trivially. If \( C \) is \( P(t_1, \ldots, t_n) \) for some \( n \)-ary predicate symbol \( P \) (possibly the logical “=”), and some terms \( t_1, \ldots, t_n \), then again the result follows trivially. For example, in the latter case we are asked to verify \( \vdash (A \leftrightarrow B) \rightarrow \Box (P(t_1, \ldots, t_n) \leftrightarrow P(t_1, \ldots, t_n)) \) which follows from 4.1 and axiom \( \Box (P(t_1, \ldots, t_n) \leftrightarrow P(t_1, \ldots, t_n)) \).

Induction steps:

1. If \( C \) is \( \neg D \) or \( D \lor G \), then the result follows by tautological implication via the obvious I.H. For example, \( \vdash \Box (D[p := A] \leftrightarrow D[p := B]) \rightarrow \Box (\neg D[p := A] \leftrightarrow \neg D[p := B]) \) by \( \Box \)-monotonicity.
   Hence \( \vdash \Box (A \leftrightarrow B) \rightarrow \Box (\neg D[p := A] \leftrightarrow \neg D[p := B]) \) by I.H. and tautological implication.

2. If \( C \) is \( (\forall x)D \), then we calculate as follows:
   \[
   \Box (A \leftrightarrow B) \\
   \rightarrow \langle \text{I.H.} \rangle \\
   \Box (D[p := A] \leftrightarrow D[p := B]) \\
   \rightarrow \langle \text{M3} \rangle \\
   \Box (\forall x)(D[p := A] \leftrightarrow D[p := B]) \\
   \rightarrow \langle 4.7 + \Box \text{-monotonicity (4.8)} \rangle \\
   \Box (\forall x)D[p := A] \leftrightarrow (\forall x)D[p := B])
   \]
   We are done since \( (\forall x)D[p := A] \) is the same as \( (\forall x)D[p := A] \).

3. If \( C \) is \( \Box D \), then we calculate as follows:
   \[
   \Box (A \leftrightarrow B) \\
   \rightarrow \langle \text{I.H.} \rangle \\
   \Box (D[p := A] \leftrightarrow D[p := B]) \\
   \rightarrow \langle \text{M2} \rangle \\
   \Box (D[p := A] \leftrightarrow D[p := B]) \\
   \rightarrow \langle 4.9 + \Box \text{-monotonicity (4.8)} \rangle \\
   \Box (\Box D[p := A] \leftrightarrow \Box D[p := B])
   \]
   We are done since \( \Box (D[p := A]) \) is the same as \( (\Box D)[p := A] \).

Corollary 4.14. \( \vdash \Box (A \leftrightarrow B) \rightarrow (\Box C[p := A] \leftrightarrow \Box C[p := B]) \).

Metatheorem 4.15. [Inner \( \forall \)-Introduction] If \( A \) has no free \( x \), then \( \vdash \Box (A \rightarrow B) \rightarrow \Box (A \rightarrow (\forall x)B) \).
PROOF.
\[ \square(A \rightarrow B) \]
\[ \rightarrow \langle \text{inner } \forall\text{-monotonicity (4.12)} \rangle \]
\[ \square((\forall x)A \rightarrow (\forall x)B) \]
\[ \leftrightarrow \langle \text{Leibniz (4.14): axioms (2, 3) yield } \square((\forall x)A \leftrightarrow A) \rangle \]
\[ \square(A \rightarrow (\forall x)B) \]

COROLLARY 4.16. [Inner \exists\text{-Introduction}] If \( B \) has no free \( x \), then
\[ \vdash \square(A \rightarrow B) \rightarrow \square((\exists x)A \rightarrow B) \]

REMARK 4.17. Each of the implications in 4.15 and 4.16 is promoted to an equivalence by tautological implication and the fact that the other direction holds. For example, \( \vdash \square(A \rightarrow B) \leftarrow \square(A \rightarrow (\forall x)B) \) by \( \square\)-monotonicity (4.8) and the tautological consequence \( (A \rightarrow (\forall x)B) \rightarrow (A \rightarrow B) \) of an instance of axiom (2).

Due to lack of space we present here only two examples of use of \( M^3 \) as a tool for writing equational proofs of classical theorems.

EXAMPLE 4.18. [\( \forall\forall \text{-swap} \)] To prove the classical \( (\forall x)(\forall y)A \) and \( (\forall y)(\forall x)A \) are mutually derivable” we prove instead \( \vdash \square(\forall x)(\forall y)A \leftrightarrow \square(\forall y)(\forall x)A \).
\[ \square((\forall x)(\forall y)A) \]
\[ \leftrightarrow \langle \text{gen (4.5)} \rangle \]
\[ \square((\forall y)A) \]
\[ \leftrightarrow \langle \text{gen} \rangle \]
\[ \square(A) \]
\[ \leftrightarrow \langle \text{gen} \rangle \]
\[ \square((\forall x)A) \]
\[ \leftrightarrow \langle \text{gen} \rangle \]
\[ \square((\forall y)(\forall x)A) \]

The classical equational proof of the above is totally disconnected since \( \vdash \) cannot be replaced by \( \rightarrow \) in the \( \forall\text{-introduction direction}. \]

EXAMPLE 4.19. What if we want the classical \( \vdash (\forall x)(\forall y)A \leftrightarrow (\forall y)(\forall x)A \) instead? We can do this by proving \( \rightarrow \) and \( \leftarrow \) directions separately, followed by tautological implication. E.g., for the \( \rightarrow \) direction we verify
\[ \vdash \Box ((\forall x)(\forall y)A \rightarrow (\forall y)(\forall x)A): \]
\[ \Box ((\forall y)A \rightarrow A) \]
\[ \rightarrow \langle \text{inner } \forall-\text{mon.} \rangle \]
\[ \Box ((\forall x)(\forall y)A \rightarrow (\forall x)A) \]
\[ \rightarrow \langle \text{inner } \forall-\text{intro.} \rangle \]
\[ \Box ((\forall x)(\forall y)A \rightarrow (\forall y)(\forall x)A) \]

References


Department of Computer Science
York University
Toronto, Ontario
M3J 1P3
Canada
gt@cs.yorku.ca

Department of Mathematics and Statistics
York University
Toronto, Ontario
M3J 1P3
Canada
francisco_kibedi@hotmail.com