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ON GLUING OF LATTICES

Abstract

We compare the operation on lattices given by A. Wroński to the operation of gluing of bounded lattices according to a skeleton introduced by Ch. Herrmann. We prove that these operations differ in many respects. In particular, sum-irreducible lattices with respect to these operations do not coincide.

The operation of gluing of lattices introduced by Wroński in [10] was a generalization of the operation considered by Troelstra in [9], which special case is the well known operation of adding the top element (the so called "mast") to a lattice (see, Jaśkowski [7]).

Let $K_1 = \langle K_1, \leq_1 \rangle$ and $K_2 = \langle K_2, \leq_2 \rangle$ be lattices such that $K_1 \cap K_2$ is a filter in $K_1$ and an ideal in $K_2$ and the orderings $\leq_1$ and $\leq_2$ coincide on $K_1 \cap K_2$. Then $K_1 \oplus K_2 = \langle K_1 \cup K_2, \leq \rangle$, where $\leq$ is the transitive closure of $\leq_1 \cup \leq_2$, is a lattice which we are going to call a W-sum of $K_1$ and $K_2$.

It is said that a lattice $K$ is W-irreducible iff there are no proper sublattices $K_1$ and $K_2$ of $K$ such that $K = K_1 \oplus K_2$.

**Theorem 1.** For every finite lattice $K$ the following conditions are equivalent:

1. $K$ is W-irreducible;
2. $\bigvee a \in K \setminus \{0\}$ $\forall N(a)$;
3. $\bigwedge a \in K \setminus \{1\}$ $\forall N(a)$.

where $N(a) = \{b \in K; a$ and $b$ are incomparable\}$ and $0, 1$ denote the smallest and the greatest element of $K$, respectively.
Proof.

1. $\Rightarrow$ 2.

Suppose that $x = \bigvee N(a) \lor a < 1$ for some $a > 0$. Let $K_1 = [0, x]$, $K_2 = [a, 1]$. Since $x \geq a$, we get $K_1 \cap K_2 = [a, x]$, which is a filter in $K_1$ and an ideal in $K_2$. Thus $K_1 \oplus K_2$ is a sublattice of the lattice $K$.

On the other hand, let $y \in K$. If $y \geq a$ then $y \in K_2$. If $y \leq a$ then $y \leq x$ and hence $y \in K_1$. In the case when $a$ and $y$ are incomparable we have $y \in K_1$. It means that $K_1 \oplus K_2 = K$ and both $K_1$ and $K_2$ are proper sublattices of $K$, which contradicts the assumption.

2. $\Rightarrow$ 1.

Suppose that $K_1 \oplus K_2 = K$, where $K_1$ and $K_2$ are proper sublattices of $K$. Then $K_1 = [0, x]$, $K_2 = [y, 1]$, for some $y \leq x < 1$. If $N(y) = \emptyset$ then $\bigvee N(y) \lor y = y < 1$. Let $a \in N(y)$. Then $a \in K_1$ and hence $a \leq x$. It yields $\bigvee N(y) \lor y \leq x < 1$.

It proves the equivalence of 1. and 2.

The proof of the equivalence of 1. and 3. is analogous. □

Corollary 1. Every finite complementary lattice is W-irreducible.

Corollary 2. A finite distributive lattice $D$ is W-irreducible iff $D$ is a Boolean lattice.

It is easy to notice that every finite lattice can be represented as a W-sum of its W-irreducible intervals. This observation together with Corollary 2 yields the following theorem, proved by Kotas, Wojtylak in [4]:

Theorem 2. Every finite distributive lattice can be represented as a W-sum of its Boolean intervals.

Every representation of a finite lattice $K$ as a W-sum of its W-irreducible intervals will be called a W-representation of $K$. There are infinitely many W-representations of a given lattice $K$ but it is obvious that all maximal W-irreducible intervals of $K$ must occur in each of them. It is not true, however, even in the distributive case, that there is always a W-representation of $K$ containing only maximal W-irreducible intervals of $K$.

If it is possible to represent a lattice $K$ as a W-sum of its maximal W-irreducible intervals without repeating components or taking their subintervals, then we call that representation a scarce W-representation (or a
scarce decomposition) of \( K \) (see [3]). In the case of a finite distributive lattice \( D \) the components of the scarce W-representation coincide with the components of the Herrmann gluing of the lattice \( D \).

The operation of gluing of lattices was introduced by Herrmann in [6] as some generalization of the algebraic operation on lattices given by Dilworth and Hall in [5]. The definition of this gluing can be formulated as follows:

Let \( K = \langle K, \leq_K \rangle \) be a bounded lattice and \( \{ L_x \}_{x \in K} \), a family of bounded lattices such that for every \( x, y \in K \):

- if \( x \prec y \) (\( y \) covers \( x \)) then \( L_x \cap L_y \neq \emptyset \);
- if \( x \leq_K y \) and \( L_x \cap L_y \neq \emptyset \) then \( L_x \cap L_y \) is a filter of \( L_x \) and an ideal of \( L_y \) and the orderings \( \leq_x \) and \( \leq_y \) coincide on \( L_x \cap L_y \);
- \( L_x \cap L_y = L_x \lor y \cap L_y \).

Then \( L = \langle \bigcup_{x \in K} L_x, \leq \rangle \), where \( \leq \) is the transitive closure of \( \bigcup_{x \in K} \leq_x \), is a lattice called \( K \)-gluing of the family \( \{ L_x \}_{x \in K} \). The lattice \( K \) is said to be the skeleton of this gluing.

It is obvious that the W-sum of two components is a special case of the Herrmann gluing, namely, the gluing with the skeleton being two-element Boolean algebra \( B_2 \). In general, W-sum of lattices can be considered as an iteration of this \( B_2 \)-gluings.

However, we are going to show that these two operations do not coincide.

Herrmann [6] proved that every finite modular lattice \( M \) is a \( K \)-gluing of its maximal atomic intervals for some lattice \( K \), which is called the skeleton of the lattice \( M \). It means, in particular, that every finite distributive lattice \( D \) is a \( K \)-gluing of its maximal Boolean intervals for some skeleton \( K \) (which will be called the skeleton of \( D \)). Hence, the components of the \( K \)-gluing and a W-representation of a finite distributive lattice coincide if the W-representation is a scarce W-representation of \( D \).

**Example 1.** There is no scarce W-representation of the free three-generated distributive lattice \( F_3(D) \) in Figure 1 (see [3]).

Thus, there is no W-representation of \( F_3(D) \) with components coinciding with the \( K \)-gluing of \( F_3(D) \). The skeleton \( K \) of \( F_3(D) \) is a diamond (see Figure 2(b)).

In the case of non-distributive lattices differences between these two operations of gluing of lattices are more noticeable.
Example 2. The lattice $\mathcal{M}$ in Figure 2(a) is modular and hence it is the $K$-gluing of five components. Figure 2(b) shows the skeleton $K$ of $\mathcal{M}$. On the other hand, by Theorem 1, $\mathcal{M}$ is $W$-irreducible.

Example 3. The lattice $\mathcal{S}$ in Figure 3 ($\mathcal{S}$ is not modular) can be represented as the $W$-sum of four components:

$$\mathcal{S} = (S_0 \oplus S_1) \oplus (S_2 \oplus S_3),$$

where $S_0 = \{0, a, b, c\}$, $S_1 = \{c, b, e, f, h\}$, $S_2 = \{a, b, d, e, g\}$, $S_3 = \{e, h, g, 1\}$.

However, there is no lattice $\mathcal{K}$ such that $\mathcal{S}$ is the $\mathcal{K}$-gluing of the family $\{S_0, S_1, S_2, S_3\}$ since the family does not fulfill the third condition of Herrmann gluing.

References


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