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ON THE EMBEDDING OF NELSON'S LOGICS*

Abstract

It will be shown that Nelson's constructive logic with strong negation **N3** can be faithfully embedded into its paraconsistent analog **N4**.

1. Introduction

In the present short note, we state the quite simple, but interesting and new fact that Nelson's constructive logic with strong negation **N3** can be embedded into its paraconsistent analog **N4**. We assume the usual convention that a logic is *paraconsistent*, when there is a contradictory, but non-trivial theory over this logic. A logic is *explosive*, when it is not paraconsistent. Recall that the logic **N3** was introduced in [4] as an alternative to the intuitionistic logic of A. Heyting. Unlike the intuitionistic logic, **N3** has a negation $\sim$ constructive in the following sense. Provability of a formula $\sim(\varphi \wedge \psi)$ implies that either $\sim \varphi$ or $\sim \psi$ are provable in **N3**. Traditionally, this last property is considered as a characterizing property of a negation to be constructive. The paraconsitent variant **N4** of Nelson's constructive logic was introduced in [1]. It is obtained by omitting the explosion axiom $\sim p \to (p \to q)$ from the list of axioms of **N3**. The choice of notation **N3** and **N4** follows [8] and is connected with Kripke-style semantics for these logics. The logic **N3** can be characterized (see, e.g., [3]) by the class of partially ordered frames with three-valued valuation functions (true, false and

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neither). The logic **N4** is characterized by the same class of frames, but valuation functions should be four-valued (true, false, neither and both) in this case. As was mentioned above we intend to find a faithful embedding of **N3** into **N4**. A similar result holds for Heyting's logic **Li** and minimal logic **Lj**, a paraconsistent analog of **Li**. In fact, in [5] a more general result was stated. For any paraconsistent extension L of minimal logic, an intuitionistic counterpart $L_{\mathbf{int}}$ was defined as $L_{\mathbf{int}} \rightleftharpoons L + \{\bot \to p\}$, and it was stated that $L_{\mathbf{int}}$ can be faithfully embedded into paraconsistent L. All these facts are illustrations of a very general situation, passing from an explosive logic to its paraconsistent analog only increases the expressive power of a logic. Recall that D. Nelson justified the introdiction of his logic with strong negation by the fact that unlike the intuitionistic logic, negative statements have non-trivial constructive content in **N3**. In his own words, this logic allows better "separation of concepts". Trying to formalize the notion of "separation of concepts" D. Nelson came to the definition of a regular translation of a logical system into its subsystem given below. He did not exclude, however, that one can find more appropriate notions in other settings. Nelson's original definition [4, p.217] looks as follows: "Let L_2 be a subsystem of L_1. By this we mean, let every formula of L_2 be also a formula of L_1, and let every provable formula of L_2 be a provable formula of L_1. In addition, let $\equiv_1$ be a specified equivalence symbol of L_1 and let $\equiv_2$ be a specified one of L_2.

Definition 1.1. A transformation * of formulas of L_1 to formulas of L_2 is said to be regular just in case:

(1) For every formula A of L_1, $A \equiv_1 A^*$ is provable in L_1.
(2) For every pair of formulas A and B of L_1, if $A^* \equiv_2 B^*$ is provable in L_2, then $A \equiv_1 B$ is provable in L_1.
(3) If E is a prime formula, then E^* is E."

The last condition was motivated by the fact that "it simply assumes that both systems are concerned with the same subject matter and start with the same basic concepts." In case of considering a paraconsistent subsystem of a logical system, this last condition looks unjustified. One of the main advantages of a paraconsistent systems is the possibility to distinguish contradictions. In this way, the class of basic concepts extends by including inconsistent concepts. Therefore, for a propositional variable p, the transformation p^* should be defined so as to distinguish a consistent fragment of p. On the other hand, all positive connectives have essentially the same sense in the paraconsistent subsystem as in the original explosive system, which leads to the restriction that the transformation * should preserve all positive connectives. Note that the latter condition immediately implies condition 2 from Definition 1.1. The transformations of this kind were defined in [5] to embed intuitionistic counterparts into paraconsistent extensions of minimal logic. A transformation of this kind arises also in this article and defines an embedding of **N3** into **N4**.

2. Preliminaries

By a *logic* we mean a set of formulae closed under the rules of substitution and *modus ponens*. We will consider a propositional language $\{\vee, \wedge, \rightarrow, \sim\}$ with the symbol $\sim$ for strong negation. An equivalence is a definable symbol, $\varphi \leftrightarrow \psi \rightleftharpoons (\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi)$. Logics will be defined via Hilbert-style deductive systems with the only rules of substitution and *modus ponens*. In this way, to define a logic it is enough to give its axioms. Nelson's logic **N3** is characterized by the following list of axioms:

A1. $p \rightarrow (q \rightarrow p)$
A2. $(p \rightarrow (q \rightarrow r)) \rightarrow ((p \rightarrow q) \rightarrow (p \rightarrow r))$
A3. $(p \wedge q) \rightarrow p$
A4. $(p \wedge q) \rightarrow q$
A5. $(p \rightarrow q) \rightarrow ((p \rightarrow r) \rightarrow (p \rightarrow (q \wedge r)))$
A6. $p \rightarrow (p \vee q)$
A7. $q \rightarrow (p \vee q)$
A8. $(p \rightarrow r) \rightarrow ((q \rightarrow r) \rightarrow ((p \vee q) \rightarrow r))$
A9. $\sim\sim p \leftrightarrow p$
A10. $\sim (p \vee q) \leftrightarrow (\sim p \wedge \sim q)$
A11. $\sim (p \wedge q) \leftrightarrow (\sim p \vee \sim q)$
A12. $\sim (p \rightarrow q) \leftrightarrow (p \wedge \sim q)$
A13. $\sim p \rightarrow (p \rightarrow q)$

Axioms A1–A12 will define the logic **N4**. The intuitionistic negation $\neg$ can be defined in **N3** as $\neg\varphi \rightleftharpoons \varphi \rightarrow \sim \varphi$. From this definition immediately follows that

$$\mathbf{N3} \vdash \sim \varphi \rightarrow \neg\varphi.$$

We say that a formula φ is in a *negative normal form*, if it contains negations only before propositional variables, in other words, if $\sim\psi$ is a subformula of φ, then ψ is a propositional variable.

DEFINITION 2.1. We define a transformation $\overline{(\cdot)}$ of formulae as follows:

1. $\overline{p} \rightleftharpoons p$, $\overline{\sim p} \rightleftharpoons \sim p$ for a propositional variable p.
2. $\overline{\sim\sim\varphi} \rightleftharpoons \overline{\varphi}$ for a formula φ.
3. $\overline{\varphi \diamond \psi} \rightleftharpoons \overline{\varphi} \diamond \overline{\psi}$, where φ and ψ are formulae and $\diamond \in \{\vee, \wedge, \rightarrow\}$.
4. For any formulae φ and ψ, $\overline{\sim(\varphi \vee \psi)} \rightleftharpoons \overline{\sim\varphi} \wedge \overline{\sim\psi}$, $\overline{\sim(\varphi \wedge \psi)} \rightleftharpoons \overline{\sim\varphi} \vee \overline{\sim\psi}$, $\overline{\sim(\varphi \rightarrow \psi)} \rightleftharpoons \overline{\varphi} \wedge \overline{\sim\psi}$.

The following assertion can be easily proved.

PROPOSITION 2.2. *For any formula φ, $\overline{\varphi}$ is in a negative normal form and* $\mathbf{N4} \vdash \varphi \leftrightarrow \overline{\varphi}$.

We have thus fixed a procedure reducing a formula to a negative normal form. Note that if φ is already in a negative normal form, then $\overline{\varphi} = \varphi$.

As was stated in [6], a semantics for **N4** can be provided by twist-structures over implicative lattices. Recall that an *implicative lattice* is an algebra $\mathcal{A} = \langle A, \vee, \wedge, \rightarrow, 1\rangle$ such that $\langle A, \vee, \wedge, 1\rangle$ is a distributive lattice with the greatest element 1 and for any $a, b \in A$, the value $a \rightarrow b$ is a pseudo-complement of a with respect to b, i.e., the supremum of the set $\{c \in A : a \wedge c \leq b\}$. We will use calligraphic letters to denote algebraic structures, the universes of structures will be denoted by corresponding italic letters. The definition below is a natural analog of twist-structures over Heyting algebras introduced by M. M. Fidel [2] and D. Vakarelov [7].

DEFINITION 2.3. Let $\mathcal{A} = \langle A, \vee, \wedge, \rightarrow, 1\rangle$ be an implicative lattice.

1. A *full twist-structure* over $\mathcal{A}$ is an algebra $\mathcal{A}^{\bowtie} = \langle A \times A, \vee, \wedge, \rightarrow, \sim\rangle$ with twist-operations defined for $(a, b), (c, d) \in A \times A$ as follows:
$$(a, b) \vee (c, d) \rightleftharpoons (a \vee c, b \wedge d),\ (a, b) \wedge (c, d) \rightleftharpoons (a \wedge c, b \vee d)$$
$$(a, b) \rightarrow (c, d) \rightleftharpoons (a \rightarrow c, a \wedge d),\ \sim(a, b) \rightleftharpoons (b, a).$$
2. A *twist-structure* over $\mathcal{A}$ is an arbitrary subalgebra $\mathcal{B}$ of the full twist-structure $\mathcal{A}^{\bowtie}$ such that $\pi_1(\mathcal{B}) = \mathcal{A}$ (in which case also $\pi_2(\mathcal{B}) = \mathcal{A}$), where π_i, $i = 1, 2$, is a projection of a direct product onto the i-th coordinate.
3. The class of all twist-structures over $\mathcal{A}$ is denoted $S^{\bowtie}(\mathcal{A})$.

A valuation in a twist-structure $\mathcal{B}$ is defined in the usual way as a homomorphism of an algebra of formulae into $\mathcal{B}$. A formula φ is *true* on a twist-structure $\mathcal{B}$, $\mathcal{B} \models \varphi$, if and only if for any valuation v in $\mathcal{B}$, we have $\pi_1 v(\varphi) = 1$. For a set of formulae Γ, the relation $\mathcal{B} \models \Gamma$ means that $\mathcal{B} \models \varphi$ for all $\varphi \in \Gamma$.

THEOREM 2.4. [6] *For any formulae φ, $\varphi \in$ **N4** if and only if $\mathcal{B} \models \varphi$ for any twist-structure $\mathcal{B}$.*

LEMMA 2.5. *Let $\mathcal{A}$ be an implicative lattice and $\mathcal{B} \in S^{\bowtie}(\mathcal{A})$. We have $\mathcal{B} \models$ **N3** if and only if $\mathcal{A}$ has the least element 0 and for any $(a, b) \in B$, $a \wedge b = 0$.*

As it follows from the definition of twist-operations, the truth of axiom A13 on $\mathcal{B}$ is equivalent to the fact that for any $(a, b), (c, d) \in B$, $a \wedge b \leq c$. By Definition 2.1, c is an arbitrary element of $\mathcal{A}$, whence, for any $(a, b) \in B$, $a \wedge b$ is the least element of $\mathcal{A}$.

LEMMA 2.6. *Let $\mathcal{A}$ be an implicative lattice with the least element 0, $\mathcal{B} \in S^{\bowtie}(\mathcal{A})$ and $\mathcal{B} \models$ **N3**. Then $b \leq a \to 0$ for any $(a, b) \in B$.*

In view of the previous lemma, we have

$$\neg(a, b) = (a \to b, a) = (a \to (a \wedge b), a) = (a \to 0, a),$$

for any $(a, b) \in B$. Thus, the desired relation is a semantical analog of an implication $\sim p \to \neg p$, which is provable in **N3**.

3. Embedding

DEFINITION 3.1. Let $\bot \rightleftharpoons \sim (p_0 \to p_0)$, where p_0 is a fixed propositional variable. We define a transformation $(\cdot)^*$ of formulae as follows:

1. For a propositional variable p,

$$(p)^* \rightleftharpoons p \vee \bot, \ (\sim p)^* \rightleftharpoons (\sim p \wedge (p \to \bot)) \vee \bot.$$

2. $(\varphi \diamond \psi)^* \rightleftharpoons \varphi^* \diamond \psi^*$, where φ and ψ are formulae in a negative normal form and $\diamond \in \{\vee, \wedge, \to\}$.
3. $(\varphi)^* \rightleftharpoons (\overline{\varphi})^*$ for any formulae φ not in a negative normal form.

THEOREM 3.2. *For any formula φ,*

1. $\varphi \in \mathbf{N3}$ *if and only if* $\varphi^* \in \mathbf{N4}$.
2. $\mathbf{N3} \vdash \varphi \leftrightarrow \varphi^*$.

PROOF. 1. First, we prove the direct implication. For any substitutional instance φ of one or another **N4**-axiom, the formula $\overline{\varphi}$ will be provable in **N4** due to Proposition 2.2. The transformation $(\cdot)^*$ preserves all positive connectives for formulae in a negative normal form, therefore, $(\overline{\varphi})^*$ is also provable in **N4** as a substituitional instance of $\overline{\varphi}$. Let $\varphi, \varphi \to \psi \in \mathbf{N3}$ and $\varphi^*, (\varphi \to \psi)^* \in \mathbf{N4}$. Note that $(\varphi \to \psi)^* = (\overline{\varphi \to \psi})^* = (\overline{\varphi} \to \overline{\psi})^* = \overline{\varphi}^* \to \overline{\psi}^*$, which immediately implies that $\psi^* = (\overline{\psi})^* \in \mathbf{N4}$. We have thus proved that the set formulae, for which the considered implication holds, is closed under *modus ponens*. In this way, it remains to check that formulae of the form

$$(\sim \varphi \to (\varphi \to \psi))^* = (\sim \varphi)^* \to (\varphi^* \to \psi^*)$$

are provable in **N4**. The latter formula is equivalent in **N4** to $((\sim \varphi)^* \wedge \varphi^*) \to \psi^*$, therefore, the desired result follows from the next lemma.

LEMMA 3.3. *For any formula φ, the following holds:*

1. $\mathbf{N4} \vdash \bot \to \varphi^*$.
2. $\mathbf{N4} \vdash \bot \leftrightarrow \varphi^* \wedge (\sim \varphi)^*$.

PROOF. 1. This item is true by definition for propositional variables and their negations. Any formula φ^* can be obtained from formulae of the form p^* and $(\sim p)^*$ with the help of positive connectives, which allows one to complete the proof by an easy induction on the structure of formulae.

2. We use again an induction on the structure of formulae. For a propositional variable p, we have

$$p^* \wedge (\sim p)^* \leftrightarrow (p \wedge \sim p \wedge (p \to \bot)) \vee \bot \leftrightarrow (\sim p \wedge p \wedge \bot) \vee \bot \leftrightarrow \bot.$$

In case of conjunction of formulae, we apply the induction hypotheses and the previous item of the lemma to obtain the following chain of equivalences:

$$(\varphi \wedge \psi)^* \wedge (\sim (\varphi \wedge \psi))^* \leftrightarrow \varphi^* \wedge \psi^* \wedge (\overline{\sim (\varphi \wedge \psi)})^* \leftrightarrow$$

$$\leftrightarrow \varphi^* \wedge \psi^* \wedge (\overline{\sim \varphi} \vee \overline{\sim \psi})^* \leftrightarrow \varphi^* \wedge \psi^* \wedge ((\sim \varphi)^* \vee (\sim \psi)^*) \leftrightarrow$$

$$\leftrightarrow ((\varphi^* \wedge (\sim \varphi)^*) \wedge \psi^*) \vee (\varphi^* \wedge (\psi^* \wedge (\sim \psi)^*)) \leftrightarrow (\bot \wedge \psi^*) \vee (\varphi^* \wedge \bot) \leftrightarrow \bot.$$

The case of disjunction can be considered similarly. For implication, we have

$$(\varphi \to \psi)^* \wedge (\sim (\varphi \to \psi))^* \leftrightarrow (\varphi^* \to \psi^*) \wedge (\overline{\sim (\varphi \to \psi)})^* \leftrightarrow$$

$$\leftrightarrow (\varphi^* \to \psi^*) \wedge (\overline{\varphi} \wedge \overline{\sim \psi})^* \leftrightarrow (\varphi^* \to \psi^*) \wedge \varphi^* \wedge (\sim \psi)^* \leftrightarrow$$

$$\leftrightarrow \varphi^* \wedge \psi^* \wedge (\sim \psi)^* \leftrightarrow \varphi^* \wedge \bot \leftrightarrow \bot.$$

So, it remains to consider the case of negation, which is trivial in view of the graphical equality $(\sim \varphi)^* \wedge (\sim\sim \varphi)^* = (\sim \varphi)^* \wedge \varphi*$.

The lemma is proved.

Now we turn to the inverse implication. Assume $\mathbf{N3} \nvdash \varphi$. Then there are a twist-structure $\mathcal{B}$, $\mathcal{B} \models \mathbf{N3}$, and its valuation v such that $\pi_1 v\varphi \neq 1$. By Lemma 2.5, $\mathcal{B} \in S^{\bowtie}(\mathcal{A})$, where $\mathcal{A}$ is an implicative lattice with the least element 0. It is clear that $v(\bot) = (0, 1)$. For any propositional variable p, $v(p) = (a, b)$, we have

$$v(p^*) = v(p) \vee v(\bot) = v(p) \vee (0, 1) = v(p),$$

$$v((\sim p)^*) = (\sim v(p) \wedge (v(p) \to (0, 1))) \vee (0, 1) = (b, a) \wedge ((a, b) \to (0, 1)) =$$

$$= (b \wedge (a \to 0), a) = (b, a) = v(\sim p_i).$$

Thus, if ψ is a propositional variable or its negation, we have $v(\psi^*) = v(\psi)$. Due to the fact that * preserve all logical connectives for formulae in negative normal form, the relation $v(\psi^*) = v(\psi)$ holds for any formula ψ in negative normal form, and so for any formula ψ by item 3 of Definition 3.1 and Proposition 2.2. In particular, we have $\pi_1 v(\varphi^*) \neq 1$, i.e., $\varphi^* \notin \mathbf{N4}$.

2. In the proof of the previous item, it was stated that for any formula φ, for any twist-structure $\mathcal{B}$, $\mathcal{B} \models \mathbf{N3}$, and any $\mathcal{B}$-valuation v, $v(\varphi^*) = v(\varphi)$, which implies $\mathbf{N3} \vdash \varphi \leftrightarrow \varphi^*$.

The theorem is proved.

We conclude this article with the following problem: to find a natural example of an explosive logic such that it is not faithfully embeddable into its paraconsistent analog. An example should be natural in a sense that the logic should have been described in the literature and not constructed especially to solve this problem.

References

[1] A. Almukdad and D.Nelson, *Constructible falsity and inexact predicates*, **Journal of Symbolic Logic** 49 (1984), pp. 231–233.

[2] M. M. Fidel, *An algebraic study of a propositional system of Nelson*, **Mathematical Logic, Proc. of the First Brasilian Conference, Campinas 1977**, Lect. Notes Pure Appl. Math. 39 (1978), pp. 99–117.

[3] M. Kracht, *On extensions of intermediate logics by strong negation*, **Journal of Philosophical Logic** 27 (1998), pp. 49–73.

[4] D. Nelson, *Constructible falsity*, **Journal of Symbolic Logic** 14 (1949), pp. 16–26.

[5] S. P. Odintsov, *Logic of classical refutability and class of extensions of minimal logic*, to appear in **Logic and Logical Philosophy** 9.

[6] S. P. Odintsov, *Algebraic semantics for paraconsistent Nelson's logic*. Submitted to **Journal of Logic and Computation**.

[7] D. Vakarelov, *Notes on N-lattices and constructive logic with strong negation*, **Studia logica** 36 (1977), pp. 109–125.

[8] H. Wansing, *Negation*, [in:] L.Goble (ed.), **The Blackwell Guide to Philosophical Logic**, Basil Blackwell Publishers, Cambridge/MA (2001), pp. 415–436.

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