Eunsuk Yang*

**Abstract**

In this paper we investigate the relevance logic T-R of the Ticket Entailment T without the reductio (R), and its extensions TE-R, TEc-R: TE-R is the T-R with the expansion (E) and TEc-R the TE-R with the chain (c). We give completeness for each T-R, TE-R, and TEc-R by using Routley-Meyer semantics.

1. Introduction

Among relevance logic systems, T of Ticket Entailment and other systems of it, briefly T-systems, have been treated with much indifference. However, “they are formally interesting and philosophically important” as Giambrone and Meyer state in [9]. First of all, the implication of T-systems itself gives us new ideas about inference or entailment: the implicational system of T \( \rightarrow \), which Anderson and Belnap first investigated in [2], was motivated as deriving from the Rylean ideas on ‘inference tickets’. It yields insight into one of deduction theorems (see [2]). It was also motivated as “entailment shorn of modality” by Anderson [1]. Thus, it gives us the idea of relevant entailment without modality. The T without the contraction (W) T \( \rightarrow \) is one among contractionless systems considered as important from early on in the study of relevance logics (see [8, 9]).

Routley and Meyer [10] investigated the semantics, so called Routley-Meyer (RM) semantics, for Positive Ticket Entailment T\(^+\) and gave its

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*I am indebted to Professor J. Michael Dunn, who was my advisor during my research visit at Indiana University, for helpful advice and guidance concerning the present paper.*
completeness. Giambrone and Meyer [9] extended its investigation to T, TW, and their extensions T∧, TW¬ with the Boolean negation ¬. They also gave completeness for these systems by using RM semantics. But logics in the neighbourhood of T, i.e., T-systems, still have not yet been enough investigated: for example, Dunn [6] investigated several relevance systems and their extensions with the chain (c), e.g., N, BN, Nc, and Bc (R-mingle (RM)), under the name “generalized Nelson logics L”, and gave completeness for these systems (but using Kripke-style semantics in place of RM one). However, he did not investigate the T with c, and thus did not do its RM semantics, even though he [4, 5] did that for RM.¹ Routley, Meyer, and Giambrone either did not do this kind of extension.

Thus, we investigate other T-systems, more exactly, the T without the reductio (R), i.e., (A → ¬A) → ¬A, T-R and its extensions TE-R, TEc-R: TE-R is the T-R with the expansion (E) below and TEc-R the TE-R with c. T-R is formally interesting in at least the following two senses: first, in the sense that it is a system between the above systems T, TW that we emphasized their importance. We can get T by adding R and W to TW. Thus, it is a subsystem of T and a supersystem of TW. Second, in the sense that T-R and its extensions above can satisfy the (generalization of) lemma of section 8.5.2 in [2]: the lemma that if A is a theorem of T→, then either A is a theorem of T→, as LC₃ does not satisfy the above postulate.

¹Since c is a theorem of RM, Dunn investigated RM without considering its postulate.
lemma in case we regard it as Heyting negation and thus take LC₃ in place of LC₃. Of course, we may think of the negation of T as an extension of Heyting negation. But it is stronger (probably too strong) than that of TW. It is formally (or algebraically) interesting how exactly to understand the negation of T, and yet its investigation is the subject of another paper.

As its first step, we investigate RM-semantics for each T-R, TE-R, and TEC-R, and thus give its completeness. This means that the T-systems taking weaker negation, i.e., quasi-negation, than that of T itself can be complete.

For convenience, by T(EC)-R we shall ambiguously express T-R, TE-R, TEC-R all together, if we do not need distinguish them, but context should determine which system is intended; often by T(E)-R, just T-R and TE-R. Depending on the works of Routley and Meyer (and Giambrone), and Dunn in [3, 5, 9, 10, 11], we can show the completeness for T(EC)-R. We shall also adopt the similar notation, terminology, and results found in them, and assume familiarity with them.

2. Axiom Schemes and Rule for T(EC)-R

For convenience, we present just the axiom schemes and the rules of inference for T(EC)-R. For the remainder we shall follow the customary notation and terminology. The formation of T(EC)-R can be given via the following list of axiom schemes and rules:

AXIOM SCHEMES

A1. A → A  
    (self-implication)
A2. (A → B) → ((B → C) → (A → C))  
    (suffixing)
A3. (A → B) → ((C → A) → (C → B))  
    (prefixing)
A4. (A → (A → B)) → (A → B)  
    (contraction)
A5. (A ∧ B) → A, (A ∧ B) → B  
    (∨-elimination)
A6. ((A → B) ∧ (A → C)) → (A → (B ∧ C))  
    (∨-introduction)
A7. A → (A ∨ B), B → (A ∨ B)  
    (∨-introduction)
A8. ((A → C) ∧ (B → C)) → ((A ∨ B) → C)  
    (∨-elimination)
A9. (A ∧ (B ∨ C)) → ((A ∧ B) ∨ (A ∧ C))  
    (distributive law)
A10. ∼∼A → A  
    (classical double negation)
A11. (A → ∼B) → (B → ∼A)  
    (contraposition)
A12. (A → B) → (A → (A → B))  
    (expansion)
A13. \((A \rightarrow B) \lor (B \rightarrow A)\) (chain)

RULES
A \rightarrow B, A \vdash B \quad \text{(modus ponens (MP))}
A, B \vdash A \land B \quad \text{(adjunction (AD))}

SYSTEMS
T-R. \ A1 - A11;
TE-R. \ T-R + A12;
TEc-R. \ TE-R + A13.

3. Routley-Meyer frames and models for T(Ec)-R

Following [3, 5, 7], calling relevant model structures Routley-Meyer (RM) frames, we define an (RM) frame. A frame is a structure \(S = (U, \sqsubseteq, R, Z, *)\), where \((U, \sqsubseteq, R, Z)\) is a left assertional frame and \(*\) is a unary operation on \(U\), such that the following definitions and postulates hold: \(\zeta \in Z\)

df2. \(\alpha \sqsubseteq \beta := \exists \zeta (R\zeta \alpha \beta)\)
df3. \(R^2\alpha \beta \gamma \delta := \exists \chi (R\alpha \beta \chi \land R\chi \gamma \delta)\)
df4. \(R^2\alpha (\beta \gamma) \delta := \exists \chi (R\alpha \chi \delta \land R\beta \gamma \chi)\)

(With respect to the following postulates, just for convenience, to represent some \(\zeta\) we take \(0\), which Routley and Meyer take in their semantics. Note that \(0\), by which we represent some \(\zeta \in Z\), itself is a member of \(Z\), i.e., \(0 \in Z\).)

p0. \(R\alpha \beta \gamma \land \alpha \tau \sqsubseteq \alpha\) imply \(R\tau \beta \gamma\) (monotonicity)
p1. \(R0\alpha \alpha\) (identity)
p2. \(R^20\alpha \beta \gamma \Rightarrow R\alpha \beta \gamma\)
p3. \(R\alpha \alpha \alpha\) (idempotence)
p4. \(R\alpha \beta \gamma \Rightarrow R^2\alpha (\alpha \beta) \gamma\)
p5. \(R^2\alpha \beta \gamma \delta \Rightarrow R^2 \beta (\alpha \gamma) \delta\)
p6. \(R^2\alpha \beta \gamma \delta \Rightarrow R^2 \alpha (\beta \gamma) \delta\)
p7. \(R\alpha \beta \gamma \Rightarrow R^2 \alpha \beta \beta \gamma\)

2Often, in proofs of sections 4, 5, by \(0\) we shall also ambiguously represent some \(\zeta\), if we do not need distinguish them, but context should determine what is intended.
p8. \[ R\alpha\beta\gamma \implies R\alpha\gamma^*\beta^* \]

p9. \[ \alpha^{**} = \alpha \]

p10. \[ R\alpha\beta\gamma \implies \exists \chi (R\chi\beta\alpha \land R\alpha\beta\gamma) \]

p11. \[ R\emptyset\alpha\beta \text{ or } R\emptyset\beta\alpha \]

For T-R, \( df_2 - df_4 \) plus \( p_0 - p_9 \);
For TE-R, \( df_2 - df_4 \) plus \( p_0 - p_{10} \);
For TEc-R, \( df_2 - df_4 \) plus \( p_0 - p_{11} \).

Note that with respect to p1 to p7 we follow the postulates in [10] rather than those in [11], since as an axiom scheme T(Ec)-R (as well as T) does not have the assertion

(1) \[ A \to ((A \to B) \to B), \]

and thus as its postulate it does not have the commutativity

(2) \[ R\alpha\beta\gamma \implies R\beta\alpha\gamma, \]

which axiom scheme and postulate R take. Note that with respect to p8 and p9 we follow the postulates in [11], and that we take p0 following [5, 7]; also that \( \sqsubseteq \) is a partial order (p.o.) on \( U \) with respect to T(E)-R and a linear order (l.o.) on \( U \) with respect to TEc-R. Following Dunn (and Hardegree) [6] (and [7]), we regard \( U \) as a set of “states of information”, and for \( \alpha, \beta \in U \), \( \alpha \sqsubseteq \beta \) means that the information of \( \alpha \) is included in that of \( \beta \).

By a model for T(Ec)-R, we mean a structure \( M = (U, \sqsubseteq, R, Z, *, \models) \), where \( (U, \sqsubseteq, R, Z, *) \) is a frame and \( \models \) is a relation from \( U \) to sentences of T(Ec)-R satisfying the following conditions:

(Atomic Hereditary Condition (AHC)) for a propositional variable \( p \), if \( \alpha \models p \) and \( \alpha \sqsubseteq \beta \), then \( \beta \models p \);
(Evaluation Clauses (EC)) for formulas \( A, B \)

\[ \land \alpha \models A \land B \text{ iff } \alpha \models A \text{ and } \alpha \models B; \]
\[ \lor \alpha \models A \lor B \text{ iff } \alpha \models A \text{ or } \alpha \models B; \]
\[ \to \alpha \models A \to B \text{ iff for all } \beta, \gamma, \text{ if } R\alpha\beta\gamma \text{ and } \beta \models A, \text{ then } \gamma \models B; \]
\[ \sim \alpha \models \sim A \text{ iff } \alpha^* \not\models A. \]

\(^3\text{Note that we may drop the postulates } p_1, p_2 \text{ since } \sqsubseteq \text{ is an information order.} \)
A formula $A$ is true on $v$ at $\alpha$ of $U$ just in case $\alpha \models A$; $A$ is verified on $M$ in case $\zeta \models A$; $A$ entails $B$ on $M$ in case $\forall \chi \in U, if \chi \models A, then \chi \models B$; $A$ $T(Ec)$-entails $B$ just in case $A$ entails $B$ in every model; and $A$ is $T(Ec)$-$R$-valid in a frame $S$ just in case it is verified in all evaluations therein. Let $\Sigma$ be the class of frames. A sentence $A$ is $T(Ec)$-$R$-valid, in symbols $\models_{T(Ec)-R} A$, if and only if $\forall \chi \in U, if \chi \models A$ then $\chi \models B$ only if $\zeta \models A \rightarrow B$. And $A$ $T(Ec)$-$R$-entails $B$ only if $A$ $T(Ec)$-$R$-valid.

4. Soundness for $T(Ec)$-$R$

Following [5], we give the soundness for $T(Ec)$-$R$. To prove it, we need the Verification Lemma below. First, by an induction on $A$, we can easily prove

**Lemma 1.** (Hereditary Condition (HC)) For any formula $A$, if $\alpha \models A$ and $\alpha \subseteq \beta$, then $\beta \models A$.

Since with respect to the connectives $\sim, \wedge, \lor, \rightarrow$, we have the same evaluations as in [11] just except that Routley and Meyer consider them to be “interpretation”, we can directly use his Lemma 1 to 3. Thus,

**Lemma 2.** (Verification Lemma) $A$ entails $B$ on $v$ only if $A \rightarrow B$ is verified, i.e., true at $\zeta \in Z$, on $v$. Thus, $A$ entails $B$ in a given model $M$, $= (U, \subseteq, R, Z, \ast, \models)$, only if $A \rightarrow B$ is $T(Ec)$-$R$-valid in the model; that is, for every $\chi \in U$ if $\chi \models A$ then $\chi \models B$ only if $\zeta \models A \rightarrow B$. And $A$ $T(Ec)$-$R$-entails $B$ only if $A \rightarrow B$ is $T(Ec)$-$R$-valid.

**Proof.** By Lemma 2 and 3 in [11] and definitions. (Using Lemma 1, we can also prove this, see the Verification Lemma in [5]).

Let $\models_{T(Ec)-R} A$ be the theoremhood of $A$ in $T(Ec)$-$R$. Then,

**Proposition 1.** (Soundness) If $\models_{T(Ec)-R} A$, then $\models_{T(Ec)-R} A$.

**Proof.** First, we prove each instance of the axiom schemes is valid in all frames, i.e., $T(Ec)$-$R$-valid. From Lemma 4 in [11], it follows that each instance of the conjunction, disjunction, and negation axiom schemes, i.e., $A5$ to $A11$, is $T(Ec)$-$R$-valid. By Lemma 5 in [10], each instance of $A1$ to $A4$ is $T(Ec)$-$R$-valid.

Thus, for $TE-R$, we need just to prove that each instance of $A12$ is $T-E-R$-valid. To do this, it suffices by Lemma 2 to assume that $\alpha \models A \rightarrow B$ and show $\alpha \models A \rightarrow (A \rightarrow B)$. To show this last, we assume that $p11$
holds, i.e., we assume that $R\alpha\beta\gamma$ and $\beta \models A$, and show that there is $\chi$ such that $R\chi\beta\alpha$ and $R\alpha\beta\gamma$ and thus $\gamma \models B$. Using $(\to)$, we obtain $\chi \models A \to (A \to B)$ from the assumptions. From this, we get $R\chi\beta\alpha$ and $R\alpha\beta\gamma$ and thus obtain $\gamma \models B$.

For TEc-R, we need just to prove that each instance of A13 is TEc-R-valid. To show this, we assume that $p_{11}$ holds and that $R\alpha\beta$ or $R\beta\alpha$ (in fact, not just $\theta$; see section 3). We need to show that either (i) if $\alpha \models A$, then $\beta \models B$, or (ii) if $\beta \models B$, then $\alpha \models A$. For (i), assume $\alpha \models A$. Since $R\alpha\beta$ by the assumptions, $\beta \models B$ by $(\to)$. The proof of (ii) is similar to (i).

Next, we need to show that the rules MP, AD preserve T(Ec)-R validity. By p3, i.e., $R\alpha\alpha\alpha$, for any $\alpha \in U$, if $\alpha \models A \to B$ and $\alpha \models A$, then $\alpha \models B$; if $\alpha \models A$ and $\alpha \models B$, then $\alpha \models A \land B$ by $(\land)$. Thus, since $0$ is a subset of $\alpha$ with respect to information, i.e., $0 \subseteq \alpha$, it is immediate that the rules MP, AD preserve T(Ec)-R-validity.

5. Completeness for T(Ec)-R

We give the completeness for T(Ec)-R by using the well-known Henkin-style proofs for modal logic, but with prime theories in place of maximal theories. To do this, we define some theories. We interpret $\vdash_{T(Ec)-R}$ as the deducibility consequence relation of the logic T(Ec)-R. By a T(Ec)-R-theory, we mean a set $T$ of sentences closed under deducibility, i.e., closed under MP and AD; by a prime T(Ec)-R-theory, a theory $T$ such that if $A \lor B \in T$, then $A \in T$ or $B \in T$; and by a trivial T(Ec)-R theory, the entire set of sentences of T(Ec)-R. As Dunn states in Remark 4 in [6], we note that a T(Ec)-R-theory $T$ contains all of the theorems of T(Ec)-R. Thus it is what has been called a “regular theory” in the relevance logic literature. That is, by a T(Ec)-R-theory we mean a regular T(Ec)-R-theory. This means that $T$ is never empty. In the results below, there is no role either for trivial T(Ec)-R theories. Hence, by a “T(Ec)-R theory” we mean a non-trivial one.

Let a canonical T(Ec)-R-frame be a structure $S = (U_{can}, \sqsubseteq_{can}, R_{can}, Z_{can}, \cdot_{can})$, where $\sqsubseteq_{can}$ is an information order on $U_{can}$, $Z_{can}$ is a set of any prime T(Ec)-R theory, i.e., $\zeta_{can} (\in Z_{can})$, $Z_{can} \subseteq U_{can}$, $U_{can}$ is the set of prime T(Ec)-R theories extending $\zeta_{can}$, $R_{can}$ is $R$ below restricted to $U_{can}$. 

\[ T-R, \ Te-R, \ Te\text{-}R \]
(3) \( R_{\alpha \beta \gamma} \) iff for any formula \( A, B \) of \( T(Ec) \)-R, if \( A \rightarrow B \in \alpha \) and \( A \in \beta \), then \( B \in \gamma \),

and \( ^*_{can} \) is \( ^* \) restricted to \( U_{can} \). We call a frame fitting for \( T(Ec) \)-R if for each axiom scheme of \( T(Ec) \)-R the corresponding semantical postulate holds. Where \( \alpha \) is a prime theory, let \( \alpha^* \) be the set of every formula \( A \) such that \( \sim A \) does not belong to \( \alpha \), i.e., \( \alpha^* = \{ A: \sim A \notin \alpha \} \).

As we mentioned above, we take the ideas of proofs from the Henkin-style completeness proofs. Thus, note that the base \( 0_{can} \), i.e., \( 0 \), among \( \zeta_{can} (\in Z_{can}) \), is constructed as a prime \( T(Ec) \)-R-theory that excludes nontheorems of \( T(Ec) \)-R, i.e., excludes \( A \) such that not \( \vdash_{T(Ec)-R} A \). Note also that in proofs below, by \( 0 \), i.e., \( 0_{can} \), we often represent \( \zeta_{can} \) (as well as \( 0 \)) if context can make clear what is intended (cf. see section 3). The partial orderedness and the linear orderedness of a canonical \( T(Ec) \)-R-frame depend on \( \subseteq \) restricted on \( U_{can} \). Then, first, it is obvious that

**Proposition 2.** A canonical \( T(Ec) \)-R-frame is partially ordered.

**Proposition 3.** For \( T(Ec) \), a canonical frame is connected (and thus linearly ordered).

**Proof.** Suppose toward contradiction that neither \( \alpha \subseteq_{can} \beta \) nor \( \beta \subseteq_{can} \alpha \). Then there is \( A \) such that \( A \in \alpha \) and \( A \notin \beta \) and there is \( B \) such that \( B \in \beta \) and \( B \notin \alpha \). Since \( \vdash_{T(Ec)-R} (A \rightarrow B) \lor (B \rightarrow A) \), then \( (A \rightarrow B) \lor (B \rightarrow A) \in 0 \) (not just \( 0 \) but also \( \zeta_{can} \)). If \( A \rightarrow B \in 0 \), then \( A \rightarrow B \in \alpha \), and thus \( B \in \alpha \) by p3 and (3), which is contrary to the supposition. The case that \( B \rightarrow A \in 0 \) is similar to the case that \( A \rightarrow B \in 0 \).

**Proposition 4.** The canonically defined \( T(Ec) \)-R-frame is a frame fitting for \( T(Ec)-R \).

**Proof.** By Theorem 1 of section 48.3 in [3], p0 holds. By Lemma 6 in [10], p1 to p7 hold. By Lemma 13 in [11], p8 and p9 hold.

Thus, to show that the canonical \( T(Ec) \)-R-frame is a frame fitting for \( T(Ec)-R \), we need just to prove that p10 holds. For p10, we assume that \( R_{\alpha \beta \gamma} \). We need to show that there is a prime theory \( \chi \) such that \( R_{\chi \beta \alpha} \) and \( R_{\alpha \beta \gamma} \). Suppose \( A \rightarrow B \in \alpha \) and \( A \in \beta \). Then, \( A \rightarrow (A \rightarrow B) \in \alpha \) by A12. Let us take \( \alpha \) to be \( \chi \). Then, by the assumptions, we obtain \( R_{\alpha \beta \alpha} \) and \( R_{\alpha \beta \gamma} \). That is, by the assumptions \( A \rightarrow (A \rightarrow B) \in \alpha \), \( A \in \beta \), we get \( A \rightarrow B \in \alpha \), and by the assumptions \( A \rightarrow B \in \alpha \), \( A \in \beta \), we obtain B.
$\gamma$, as desired. Since $\alpha$ is a prime theory, it ensures that there is a prime theory that satisfies $p10$.

To show that the canonical TEc-R-frame is a frame fitting for TEc-R, we need to prove that $p11$ holds. For $p11$, we assume that $A \rightarrow B \in 0$ or $B \rightarrow A \in 0$ (not just 0 but also $\zeta_{can}$) by A13 and (v). We need to show that $R0\alpha\beta$ or $R0\beta\alpha$, i.e., that (i) if $A \in \alpha$ then $B \in \beta$ or (ii) if $B \in \beta$ then $A \in \alpha$. For (i), assume $A \in \alpha$. Since $R0\alpha\beta$ by the assumptions, $B \in \beta$ by $(\rightarrow)$. The proof of (ii) is similar to (i).

Next, we need to define an appropriate relation $|= on S = (U_{can}, \subseteq_{can}, R_{can}, Z_{can}, *_{can})$. We define it to be that $\alpha |= A$ iff $A \in \alpha$.

However, we need to verify that this satisfies AHC and EC above. Note that since the positive part of T(Ec)-R satisfies Definition 1 of section 42.1 in [3], we can directly use Fact 1 and Fact 2 of section 48.3 in [3], which are considered for $R^+$, and thus we can use Theorem 2 of the same section.

**Proposition 5.** The canonically defined $(U_{can}, \subseteq_{can}, R_{can}, Z_{can}, *_{can}, |=)$ is indeed a T(Ec)-R model.

**Proof.** For AHC, i.e., the Atomic Hereditary Condition, suppose that $\alpha \subseteq \beta$, i.e., $R0\alpha\beta$, and that $p \in \alpha$. Then, $p \in \beta$ by df2.

For EC, i.e., the Evaluation Clauses, first, the clauses $(\land), (\lor)$ are immediate. Thus, we prove the clauses $(\rightarrow), (\sim)$. By applying the canonical definition of $|=\$, the clause $(\rightarrow)$ amounts to

$(\rightarrow_c) \quad A \rightarrow B \in \alpha$ iff for all $\beta, \gamma$, if $R0\alpha\beta$ and $A \in \beta$, then $B \in \gamma$.

This is by Theorem 2 of section 48.3 in [3], as we noted above. That the canonical definition of $|=\$ satisfies $(\sim)$ is immediate, just by using the double negation; that is, let $A = \sim B$. We must show that $\sim B \in \alpha$ iff $B \notin \alpha^\ast$. It is obvious since $\sim B \in \alpha$ iff $\sim B \notin \alpha^\ast$, i.e., $\sim B \notin \{C: \sim C \notin \alpha\}$, iff $B \notin \alpha^\ast$.

Thus, $(U_{can}, \subseteq_{can}, R_{can}, Z_{can}, *_{can}, |=)$ is a T(Ec)-R model. So, since, by construction, 0 excludes our chosen nontheorem $A$ and the canonical definition of $|=\$ agrees with membership, we can state that for each non-theorem $A$ of T(Ec)-R, there is a T(Ec)-R model $A$ in which $A$ is not $0 |= A$. It gives us the (weak) completeness for T(Ec)-R as follows.
Theorem 1. (Weak Completeness) If $\models_{T(Ec)-R} A$, then $\vdash_{T(Ec)-R} A$.

Next, let us prove the strong completeness for $T(Ec)$-R. As $R^+$ in [3], we define $A$ to be a $T(Ec)$-R consequence of a set of formulas $\Gamma$ if and only if for every $T(Ec)$-R model, whenever $\alpha \models B$ for every $B \in \Gamma$, $\alpha \models A$, for (not just 0 but) all $\alpha \in U$. Let us say that $A$ is $T(Ec)$-R deducible from $\Gamma$ if and only if $A$ is in every $T(Ec)$-R theory containing $\Gamma$. Then,

Proposition 6. If $\Gamma \not\vdash_{T(Ec)-R} A$, then there is a prime theory $\zeta$ such that $\Gamma \subseteq \zeta$ and $A \notin \zeta$.

Proof. Take an enumeration $\{A_n: n \in \omega\}$ of the well-formed formulas of $T(Ec)$-R. We define a sequence of sets by induction as follows:

\[
\begin{align*}
\zeta_0 &= \{Ar: \Gamma \vdash_{T(Ec)-R} A\} \\
\zeta_{i+1} &= \text{Th}(\zeta_i \cup \{A_{i+1}\}) & \text{if it is not the case that} \ zeta_i, A_{i+1} \vdash_{T(Ec)-R} A, \\
\zeta_i &= \text{Th}(\zeta_i) & \text{otherwise.}
\end{align*}
\]

Let $\zeta$ be the union of all these $\zeta_n$'s. It is easy to see that $\zeta$ is a theory not containing $A$. Also we can show that it is a prime.

Suppose toward contradiction that $B \lor C \in \zeta$ and $B, C \notin \zeta$. Then the theories obtained from $\zeta \cup B$ and $\zeta \cup C$ must both contain $A$. It follows that there is a conjunction of members of $\zeta \cup B$ such that $\zeta \cup B \vdash_{T(Ec)-R} A$ and $\zeta \cup C \vdash_{T(Ec)-R} A$. Note that if $\vdash_{T(Ec)-R} A \rightarrow B$, then $A \vdash_{T(Ec)-R} B$. Then, by A8 and MP, we get $\zeta \vdash B \lor (\zeta \land C)$ and $\zeta \vdash (\zeta \land B) \vdash_{T(Ec)-R} A$. And we obtain $\zeta \land (B \lor C) \vdash_{T(Ec)-R} A$ by A3, A9, and MP. From this we get that $A \in \zeta$, which is contrary to our supposition. Thus, by using Propositions 5 and 6 we can show its strong completeness as follows.

Theorem 2. (Strong Completeness) If $\Gamma \models_{T(Ec)-R} A$, then $\Gamma \vdash_{T(Ec)-R} A$.

References


Institute of Yonsei Philosophy
Yonsei University
Rm 523 Inmun-gwan
Seodaemun Seoul 120-789
Korea
e-mail:yanges2@hanmail.net