A VIEW OF LOGICAL OMNISCIENCE PROBLEM

Abstract

We consider a mathematical approach to study the logical omniscience problem. First, the property of a formula of being a consequence of a set of formulas is analysed (as it seems to give rise to some subtle problems). Next, we study axiomatic logic systems of agents and their knowledge in the light of the logical omniscience problem. We find necessary and sufficient conditions for an agent to rejoice logical omniscience for several distinct cases of models classes and logical consequences, in particular, for first order logic models and Kripke-like models for logics of knowledge. This conditions show, in particular, how to avoid logical omniscience if we are basing on axiomatic approach to knowledge of agents.

KEY WORDS: logical omniscience problem, inference rule, first-order logic, Kripke model

1. Introduction, Initial Conventions and Simple Observations

The motivation for this paper is a recent research on the problem of logical omniscience and common knowledge nowadays (cf., for instance, [1], [3], [4], [5] and [7]), and the question how to avoid full logical omniscience. In the beginning we do not specify the meaning of the fact that a formula $\alpha$ is a consequence of a set of formulas $\Gamma$ on a class of models $K$ (we fix notation

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\( \Gamma \Rightarrow_{K} \alpha \) for this kind consequence). It could be a standard consequence in the classical first order logic, or the consequence in Kripke models etc. For the time being it is not formal and will be specified in the sequel.

An agent \( A_i \) is fully logically omniscient (is \( \text{FLO} \)) about a class of models \( K \) iff for any set of formulas \( \Gamma \) and a formula \( \varphi \), \( \Gamma \Rightarrow_K \varphi \) implies \( A_i \) knows that \( \varphi \) follows from all formulas \( \Gamma \). We will use notation \( \Gamma \Rightarrow_{A_i} \varphi \) to render this fact.

Yet we have not explained what \textit{knows} means. Initially, we may determine an axiomatic system \( A_{x_i} \) to imitate knowledge of \( A_i \) and then treat it as a basis for defining knowledge about logical consequence, possibly in manifold ways.

**Definition 1.1.** We say

- (i) \( A_i \) strongly knows that \( \alpha \) is a logical consequence of \( \Gamma \) (denotation \( \Gamma \Rightarrow_{A_i,s} \alpha \)) iff, for some finite subset \( \Gamma_1 \) of \( \Gamma \), \( \Gamma \models_{A_{x_i}} \Gamma_1 \rightarrow \alpha \).
- (ii) \( A_i \) knows that \( \alpha \) is a logical consequence of \( \Gamma \) (denotation \( \Gamma \Rightarrow_{A_i} \alpha \)) iff \( \Gamma \models_{A_{x_i}} \alpha \).
- (iii) For an axiomatic system \( A_{x_i} \) to be a propositional logic, \( A_i \) globally knows that \( \alpha \) is a logical consequence of \( \Gamma \) (denotation \( \Gamma \Rightarrow_{A_i,g} \alpha \)) iff, for any substitution \( s \) of formulas for variables, \( \Gamma \models_{A_{x_i}} \Gamma^s \) implies \( \Gamma \models_{A_{x_i}} \alpha^s \).

It is evident that, for any \( A_{x_i} \) with a standard definition of \( \Gamma \Rightarrow_{A_{x_i}} \) (to be a consequence by a finite number of applications of axioms and inference rules), for which the modus ponens rule \( x, x \rightarrow y/y \) is admissible, the following holds: \( [\Gamma \Rightarrow_{A_{x_i},s} \alpha] \Rightarrow [\Gamma \Rightarrow_{A_{x_i}} \alpha] \Rightarrow [\Gamma \Rightarrow_{A_{x_i},g} \alpha] \).

Though nowhere above we have coincidence. Indeed, for any structurally incomplete logic \( A_{x_i} \) (cf. for examples [8], for instance) \( \Gamma \Rightarrow_{A_{x_i},g} \alpha \) does not imply \( \Gamma \Rightarrow_{A_{x_i},s} \alpha \) for some \( \Gamma \) and \( \alpha \). And \( \Gamma \Rightarrow_{A_{x_i}} \alpha \) does not always imply \( \Gamma \Rightarrow_{A_{x_i},s} \alpha \) (take as an example \( A_{x_1} \) which is non-trivial normal modal logic).

Now attempting to define \( \Gamma \Rightarrow_{K} \alpha \) wherein \( \alpha \) is a consequence of formulas \( \Gamma \) in a class of first order models \( K \), we suggest the following:

**Definition 1.2.** For a class of models \( K \), a set of formulas \( \Gamma \) and a formula \( \alpha \),

- \( \Gamma \Rightarrow_{\text{int}} K \alpha \) (\( \alpha \) is an interior consequence of \( \Gamma \) in \( K \)) iff
∀A ∈ K[∀ϕ ∈ Γ(A |= ϕ)]⇒A |= α

• Γ⇒\text{ext}_K α (α is an external consequence of Γ in K) iff

∀A ∈ K, ∀ϕ ∈ Γ(A |= ϕ)]⇒∀A ∈ K, A |= α].

As for as the essence of validness of formulas on models is concerned notice that Γ⇒\text{int}_K α always implies Γ⇒\text{ext}_K α. But the converse is not valid in general. Indeed, let ⊥ be the formula ¬(p → p). It is trivial to derive that: For arbitrary class of models K and any set of formulas Γ and a given formula α, [Γ⇒\text{ext}_K α]⇔[Γ⇒\text{int}_K α] ∨ [Γ⇒\text{ext}_⊥].

Therefore internal and external consequences do not differ much; the only point is that the external consequence can express the fact that the formulas of the premise can be disproved by at least one particular model from the class K. However, this is interesting because it allows to state, say, the independence of mathematical properties which can be expressed by formulas.

2. Observations on Logical Consequence in Propositional Logic

If we now turn to analyze internal and external consequences for propositional logics then there is more diversity in their interpretations. For algebraic semantics (when we consider classes of algebras from varieties Var(L) of algebras in the signature of these logics treating validness of formulas as validity of special identities) the notions of internal and external consequence will be, generally speaking, within the approach given above for the case where we consider classes of algebraic systems and restrict first-order formulas to the case of identities and nothing new will appear. The classes of Kripke frames and Kripke models bring about serious distinction, discussed below. First we assume K to be a class of Kripke frames which are sets with binary accessibility relations. The number of accessibility relations R_1, ..., R_n corresponds to the number of modal-like operations in the language of propositional logic L which is assumed to be chosen and fixed. For any formula ϕ in the language of L and for any valuation V of its variables in any M ∈ K, there is a definition of validity of
φ under V in any element a of M (the notation is \((M, a) \Vdash \varphi\), where we omit M if its essence is clear from context). These definitions are similar but depend on the class of logic considered (see [2], [3], [4], [8]). \(M \models \varphi\) means that \(\varphi\) is valid in all elements of M under any valuation. Thus temporal, modal, superintuitionistic logics, logics of believe, knowledge and common knowledge are within our consideration. Below we will consider substitutions \(s\) of formulas for variables of formulas, the notation is \(s \in Sub\). The result of the substitution \(s\) in a formula \(\alpha\), in what follows, is denoted by \(\alpha^s\), the same concerns a set of formulas Γ. As the notion of the consequence for a class \(K\) of Kripke frames we suggest the following:

**Definition 2.1.** For a set of formulas Γ and a given formula \(\alpha\),

- \(\Gamma \vdash^K \alpha\) (\(\alpha\) is an interior consequence of Γ in \(K\)) iff
  
  \[(\forall a \in M, \forall V \in Val(M), \forall \varphi \in \Gamma, (M, a) \Vdash \varphi \implies (M, a) \Vdash \alpha)\]

- \(\Gamma \vdash^K \alpha\) (\(\alpha\) is weakly external consequence of Γ in \(K\)) iff
  
  \[(\forall a \in M, \forall V \in Val(M), \forall \varphi \in \Gamma, (M, a) \Vdash \varphi \implies (\forall \varphi \in \Gamma) ((M, a) \Vdash \alpha))\]

- \(\Gamma \vdash^K \alpha\) (\(\alpha\) is strong external consequence of Γ in \(K\)) iff
  
  \[(\forall s \in Sub, \forall a \in M, \forall V \in Val(M), (M, a) \Vdash \Gamma^s \implies (\forall a \in M, (M, a) \Vdash \alpha^s))\]

**Lemma 2.2.** \(\Gamma \vdash^{K} \alpha\) \(\implies\) \(\Gamma \vdash^{K} \alpha\) \(\implies\) \(\Gamma \vdash^{K} \alpha\).

**Proof.** The first implication is evident. To show the second one, assume \(s \in Sub\) and \(\forall M \in K, \forall V \in Val(M), (M, a) \Vdash \Gamma^s\). Since \(\Gamma \vdash^{K} \alpha\) and, for any model \(M \in K\), for any valuation \(V \in Val(M)\) and
for any element \( a \in \mathcal{M} \), \( [(\mathcal{M}, a)]_V \models \Gamma^* \), we immediately see that for all \( \mathcal{M} \in K \), \( \forall a \in \mathcal{M}, [(\mathcal{M}, a)]_V \models \Gamma^* \) holds for any valuation \( V \). Since this holds for all \( s \in \text{Sub} \), we conclude that \( \Gamma \Rightarrow_{s,\text{ext}} \alpha \). ■

To prove the converse of the first arrow of this lemma it is sufficient to take \( p \Rightarrow_{w,\text{ext}}^{K_M} \top \), where \( KM \) is the class of all Kripke frames for modal propositional logics. The converse of the second arrow can be disproved by admissible but not derivable rule \( \Diamond p \land \Diamond \neg p/q \) of the modal system \( S4 \) and the class of all reflexive and transitive Kripke frames.

**Lemma 2.3.** For \( PS \) to be the class of all finite partially ordered sets, and intuitionistic formulas \( \neg p \rightarrow q \lor r \) and \( (\neg p \rightarrow q) \lor (\neg p \rightarrow q) \)

\[
\neg p \rightarrow q \lor r \Rightarrow_{s,\text{ext}} PS (\neg p \rightarrow q) \lor (\neg p \rightarrow q) \text{ but }
\]

\[
\neg p \rightarrow q \lor r \not\Rightarrow_{w,\text{ext}} PS (\neg p \rightarrow q) \lor (\neg p \rightarrow q) \text{ and }
\]

\[
\neg p \rightarrow q \lor r \not\Rightarrow_{s,\text{ext}} \bot.
\]

**Proof.** First and second assertions hold because

\[
\neg p \rightarrow q \lor r
\]

\[
(\neg p \rightarrow q) \lor (\neg p \rightarrow q)
\]

is an admissible but not derivable rule of the intuitionistic logic as it was shown by Harrop (cf. [8]). Again it is trivial to see that the premise of this rule can be satisfied in a suitable Kripke model, therefore the last claim of this lemma holds, too. ■

Thus no way as before to characterize strong external consequence by means of even weak external consequence. Though for to characterize weak external consequence still there is a way.

**Lemma 2.4.** For arbitrary class of Kripke models \( K \) and any finite set of formulas \( \Gamma \) and arbitrary given formula \( \alpha \),

\[
[\Gamma \Rightarrow_{w,\text{ext}} ^M \alpha] \iff \forall \mathcal{M} \in K [(\Gamma \Rightarrow_{\text{int}} ^M \alpha) \lor (\Gamma \Rightarrow_{w,\text{ext}} ^M \bot)].
\]
Proof. \( \Rightarrow \) Assume that a model \( M \in \mathcal{K} \) is given and that first \( [\Gamma \Rightarrow \mathcal{M} \, w.ex] \perp \) holds. Then the premise \( \Gamma \) fails in \( M \) for any valuation and this model \( M \in \mathcal{K} \) cannot be an obstacle for weak external consequence. If \( [\Gamma \Rightarrow \mathcal{M} \, w.ex] \alpha \) holds we conclude: \( [\Gamma \Rightarrow \mathcal{M} \, w.ex] \alpha \). Since \( M \in \mathcal{K} \) is arbitrarily, chosen \( [\Gamma \Rightarrow \mathcal{K} \, w.ex] \alpha \) holds.

Conversely, assume that \( [\Gamma \Rightarrow \mathcal{K} \, w.ex] \alpha] \). Take an arbitrary model \( M \in \mathcal{K} \). If \( \forall a \in M, \forall \beta \in \Gamma(M, a) \models \neg \beta \) our assumption \( [\Gamma \Rightarrow \mathcal{K} \, w.ex] \alpha \) gives us \( \forall a \in M(M, a) \models \forall \beta \in \Gamma(M, a) \models \neg \beta \Rightarrow \Gamma(M, a) \models \forall \beta \alpha \). In particular, \( \forall a \in M(\models \forall \beta \in \Gamma(M, a) \models \neg \beta \Rightarrow \Gamma(M, a) \models \forall \beta \alpha \), i.e. \( [\Gamma \Rightarrow \mathcal{M} \, int] \alpha \). If there is \( a \in M \) such that \( \forall \beta \in \Gamma(M, a) \models \neg \beta \) fails then \( [\Gamma \Rightarrow \mathcal{M} \, w.ex] \perp \) which is what we needed.

Let us briefly pause to discuss one more possible treatment of the notion of consequence in the classes of Kripke frames. We have not mentioned yet the following: \( [\Gamma \Rightarrow \mathcal{K} \, ext] \alpha \iff \forall M \in \mathcal{K} (M \models [\Gamma \Rightarrow \mathcal{M} \, \forall \beta \alpha]) \), where \( \models \) means the validity of formulas of all valuations. This is a degenerated case, say, for a finite set of formulas \( \Gamma \), \( [\Gamma \Rightarrow \mathcal{K} \, ext] \alpha \iff [(\alpha \in L(K)) \lor (\bigwedge \Gamma \not\in L(K))] \). Here nothing can be said on the relations and dependence between \( \Gamma \) and \( \alpha \).

Now after these simple observations on logical consequence we will investigate the knowledge of agents about logical consequence with the aim to describe the logical omniscience effect.

3. Logical Omniscience

Combining our conventions about logical consequence and an axiomatic approach to the knowledge of agents we are going to analyze the effect of logical omniscience applying the tools of mathematical logic. The axiomatic approach to the knowledge of agents will be presented being meaningful and corresponding to our intuitive claim that agent’s knowledge is generated by some facts evident for him, i.e. axioms, and some collection set of instruments, permissible from his viewpoint i.e. inference rules used to enlarge his knowledge on the basis of axioms. As soon as such an axiomatic system \( \mathcal{A}_i \), is given, we are quite free to define, within this system, what an agent \( \mathcal{A}_i \) knows, and due to this distinct approaches to logical omniscience appear. To illustrate this, we begin with the case of classes of first order models (algebraic systems), and agents axiomatic systems as certain classical first order theories.
DEFINITION 3.1. Let $A_i$ be an agent, and $Ax_i$ an axiomatic system of $A_i$.

- An agent $A_i$ is fully logically omniscient about a class of first-order models $K$ (notation is $A_i \in FLO_1(K)$) iff, for any set of formulas $\Gamma$ and any formula $\varphi$, $[\Gamma \Rightarrow_{int} K \varphi] \Rightarrow [\Gamma \vdash_{Ax_i} \varphi]$ (i.e. $[\Gamma \Rightarrow_{Ax_i} \varphi]$)

- An agent $A_i$ is globally logically omniscient about a class of first-order models $K$ (notation is $A_i \in GLO_1(K)$) iff, for any set of formulas $\Gamma$ and any given formula $\varphi$, $[\Gamma \Rightarrow_{ext} K \varphi] \Rightarrow [\Gamma \vdash_{Ax_i} \varphi]$ (i.e. $[\Gamma \Rightarrow_{Ax_i} \varphi]$)

The research in Artificial Intelligence provides a wide discussion of the effect of logical omniscience as a non-avoidable placebo effect of formal approach to the definitions of agents knowledge as well as to ways how to avoid it. Primarily it is connected with the constraints put on derivations in $Ax_i$. First we have to offer a very simple necessary and sufficient condition for full logical omniscience in classes of first order models. In what follows $Th(Ax_i)$ denotes the set of formulas provable in $Ax_i$.

LEMMA 3.2. $A_i \in FLO_1(K) \iff Th(K) \subseteq Th(Ax_i)$.

PROOF. ($\Rightarrow$) Assume that $\varphi \in Th(K)$ holds. Then $\emptyset \Rightarrow_{int} K \varphi$ and due to $A_i \in FLO_1(K)$ we conclude: $\vdash_{Ax_i} \varphi$, i.e. $\varphi \in Th(Ax_i)$.

($\Leftarrow$) Assume that a set of formulas $\Gamma$ and a formula $\varphi$ are given and $Th(K) \subseteq Th(Ax_i)$. Suppose $[\Gamma \Rightarrow_{int} K \varphi]$. Then, for any model $M \in K$, $M \models \Gamma \Rightarrow M \models \varphi$. Consequently, by compactness theorem, for some finite $\Gamma_1 \subseteq \Gamma$, $\bigwedge \Gamma_1 \Rightarrow \varphi \in Th(K) \subseteq Th(Ax_i)$. Therefore, $\Gamma \vdash_{Ax_i} \varphi$ which is what we need.

EXAMPLE 1. If $Ax_i$ is $PA$, the Peano axiomatic system for the arithmetic, and $K := \{N\}$ is the standard model for arithmetic, then $PA$ is not $FLO_1$ about $N$ since $Th(N) \not\subseteq Th(PA)$ due to Goedel incompleteness theorem. Moreover, since $Th(N)$ is complete and undecidable there are no $FLO_1$ agents for $N$ with a recursive axiomatic system.

EXAMPLE 2. If $K$ is the class of all unbound, linear and dense order sets then $K$ has the well known finite axiomatization $DLO$ which is complete for $K$ and $DLO$ is $FLO_1$ about $K$.

The immediate observation from the above very simple lemma is that in order to be $FLO_1$ it is sufficient to be aware only of valid formulas but not of their consequences (if we are within classical first order logic and classes of first-order models). And if we do not abandon the axiomatic
approach to the knowledge of agents in classical first order logic the only way to avoid $FLO_1$ is to make our axiomatic systems $Ax_i$ weaker than the theories of models under consideration.

Let $Th$ be any first-order theory, $Ad_0(Th)$ is the set of all sequents $\Gamma/\varphi$ such that $\Gamma \subseteq Th$ implies $\varphi \in Th$, so it is the so called admissible rule without variables.

**Lemma 3.3.** $Ax_i \in GLO_1(K)$ if and only if $[(\forall \Gamma/\varphi \in Ad_0(Th(K))) (\Gamma \vdash_{Ax_i} \varphi) \ 	ext{and} \ Th(K) \subseteq Th(Ax_i)]$.

**Proof.** Assume $Ax_i \in GLO_1(K)$. Suppose that the elementary first-order theory $Th(K)$ is closed w.r.t the rule $\Gamma/\varphi$, the rule which has no entry points - no variables which could be replaced by formulas, i.e. $\Gamma/\varphi \in Ad_0(Th(K))$. Then it immediately follows that $[\Gamma \Rightarrow_{ext} \varphi]$. Because $Ax_i \in GLO_1(K)$ we conclude $[\Gamma \vdash_{Ax_i} \varphi]$. Since $\Gamma$ may be, in particular, empty it follows $Th(K) \subseteq Th(Ax_i)$. So, $\Rightarrow$ holds. The converse, $\Leftarrow$, is evident.

Using this observation we see that there are many ways to avoid $GLO_1$ and not to make axiomatic systems of agents strongly weaker than the theory of classes of models.

**Example 3.** Take $K$ to be a class of all partially ordered sets and $\Gamma$ to be the collection of formulas saying that the set is linearly ordered and $\varphi$ to be the formula saying that the set is bound. Then this pair shows that even $Th(K)$ is not $GLO_1$ w.r.t. $K$. So, $GLO_1(K)$ is strongly weaker than $FLO_1(K)$ for the first order theories.

To analyze logical omniscience for propositional logics w.r.t. classes of Kripke frames we first introduce the following definition.

**Definition 3.4.** We say,

- Agent $Ax_i$ is fully logically omniscient about $K$ (notation is $Ax_i \in FLO_p(K)$) iff, for any finite set of formulas $\Gamma$ and any formula $\varphi$, $[\Gamma \Rightarrow_{int} \varphi] \Rightarrow [\Gamma \vdash_{Ax_i} \varphi]$ (i.e. $[\Gamma \Rightarrow_{Ax_i} \varphi]$).

- Agent $Ax_i$ is globally logically omniscient w.r.t. $K$ ($Ax_i \in GLO_p(K)$) if and only if, for any finite set of formulas $\Gamma$ and any formula $\varphi$, the following holds : $[\Gamma \Rightarrow_{ext} \varphi] \Rightarrow [\Gamma \vdash_{Ax_i} \varphi]$ (i.e. $[\Gamma \Rightarrow_{Ax_i} \varphi]$).
So, generally speaking, following the above scheme for first-order logics and propositional logics allows us making comprehensive description of logical omniscience. Recall that $\Gamma/\varphi \in \text{Ad}(L)$ for a logic $L$ means that $\Gamma/\varphi$ is an admissible rule for $L$, i.e. $L$ is closed w.r.t. this rule.

**Lemma 3.5.** The following holds:

(i) $A_i \in \text{FLO}_p(K) \iff L(K) \subseteq L(Ax_i)$.

(ii) $A_i \in \text{GLO}_p(K) \iff [\forall (\Gamma/\varphi \in \text{Ad}(L(K))) (\Gamma \vdash Ax_i \varphi) \land L(K) \subseteq L(Ax_i)]$.

**Proof.** (i): Assume $A_i \in \text{FLO}_p(K)$. First suppose that $\varphi \in L(K)$. Hence, $\emptyset \Rightarrow L_{\text{int}} \varphi$ and due to $A_i \in \text{FLO}_p(K)$ we have $\vdash_{Ax_i} \varphi$, and thus $\varphi \in L(Ax_i)$. For the converse, assume $L(K) \subseteq L(Ax_i)$. Let us consider a set of formulas $\Gamma$ and a formula $\varphi$. Assume $[\Gamma \Rightarrow L_{\text{int}} \varphi]$. Then, for any model $M \in K$ and any element $a$ of $M$, then $M, a$ implies $\vdash_{s \text{-ext}} \Gamma \Rightarrow (M, a) \vdash_{s \text{-ext}} \varphi$. Therefore $\bigwedge \Gamma \Rightarrow \varphi \in L(K) \subseteq L(Ax_i)$. And with modus ponens being a rule of $Ax_i$, $\Gamma \vdash_{Ax_i} \varphi$. Thus $A_i \in \text{FLO}_p(K)$.

(ii): Assume that the right part of (ii) holds. And suppose $[\Gamma \Rightarrow L_{s \text{-ext}} \varphi]$. Then, for any substitution $s$ of formulas for variables, $K \vdash_{s \text{-ext}} \varphi^s$, i.e. $\Gamma/\varphi \in \text{Ad}(L(K))$ and given the right part of (ii) we conclude: $\Gamma \vdash_{Ax_i} \varphi$. Thus $Ax_i \in \text{GLO}_p(K)$. Conversely, let $Ax_i \in \text{GLO}_p(K)$. First assume $\Gamma/\varphi \in \text{Ad}(L(K))$ then it immediately follows $\Gamma \Rightarrow L_{s \text{-ext}} \varphi^s$ and applying $Ax_i \in \text{GLO}_p(K)$ we get $\Gamma \vdash_{Ax_i} \varphi$. To show the last part, if $\varphi \in L(K)$ then $\emptyset \Rightarrow L_{s \text{-ext}} \varphi^s$, and in view of $Ax_i \in \text{GLO}_p(K)$ it follows $\vdash_{Ax_i} \varphi$. i.e. $\varphi \in L(Ax_i)$. ■

Clearly, in the second part of (ii) the last assertion follows from the first one, and we have put it in this form only in order to emphasize the including. Again it is visible that if the axiomatic version of $Ax_i$ agent’s knowledge is presented, there is no way to avoid $\text{FLO}_p$ but to choose its axiomatic system not stronger than the logic of $K$.

In turn, there are many ways to avoid $\text{GLO}_p$, due to the above lemma, in particular we can choose an $Ax_i$ which is not structurally complete w.r.t. admissible rules of $L(K)$. Take, as an example, $K$ to be all reflexive and transitive frames and $Ax_i$ to be modal logic $S4$, $S4$ is not $\text{GLO}_p$ w.r.t. $K$ though it axiomatizes it ($S4$ is not structurally complete: there are rules which are admissible but not derivable in $S4$. cf. [8])).
To develop further our tools consider the meaning of an agent knows. The meaning $\vdash_{Ax_i} \varphi$ as $i$-th agent knows $\varphi$ is reasonable, which means he knows $\varphi$ iff $\varphi$ is provable in his axiomatic system. But still we could distinguish what is provable in the agent’s axiomatic system $Ax_i$ and what he knows. For this we could, in particular, consider special formulas with a propositional letter $x$, $Kn_i(x)$, from the language of $Ax_i$ with informal meaning: $i$-th agent knows $x$ iff $Kn_i(x)$ is provable in $Ax_i$. This step improves previous approach at least in four respects:

- The meaning of what agent knows about $x$ is specified mathematically: he knows only those things which formula $Kn_i(x)$ encodes.
- We avoid uncertain and too global meaning $Ax_i$ knows $x$ as $Ax_i$ knows everything about $x$, instead we specify and restrict the agents knowledge.
- It is more modest to say that an agent knows some particular information about $x$ but not everything what could be proved about $x$. And it decreases the effect of logical omniscience about provable facts.
- We do not abandon completely previous approach being free to choose $Kn_i(x)$ to be simply $x$.

This gives rise to the analysis of logical omniscience in the light of our above definition for an agent knowledge. We begin with the case of propositional logics assuming the formula $Kn_i(x)$ to be fixed.

**Definition 3.6.** Let $A_i$ be an agent, $Ax_i$ be an axiomatic system of $A_i$.

- An agent $A_i$ is fully logically omniscient about a class of Kripke frames $\mathcal{K}$ (notation is $A_i \in FLO_{Kn,p}(\mathcal{K})$) iff, for any finite set of formulas $\Gamma$ and any given formula $\varphi$,

  $$(\Gamma \models_{int} \varphi) \Rightarrow (\Gamma \vdash_{Ax_i} Kn_i(\varphi))$$

- An agent $A_i$ is globally logically omniscient about a class of Kripke frames $\mathcal{K}$ (notation is $A_i \in GLO_{Kn,p}(\mathcal{K})$) iff, for any finite set of formulas $\Gamma$ and any given formula $\varphi$,

  $$(\Gamma \models_{s.ext} \varphi) \Rightarrow (\Gamma \vdash_{Ax_i} Kn_i(\varphi))$$
We say the formula \( Kn_i(x) \) is inwardly monotone in \( L(\mathcal{A}_x) \) if and only if \( Kn_i(x \to y) \vdash_{\mathcal{A}_x} Kn_i(x) \to Kn_i(y) \). For any set \( \mathcal{X} \) of formulas, \( Kn_i(\mathcal{X}) \) is the set \( \{Kn_i(\psi) \mid \psi \in \mathcal{X} \} \).

**Theorem 3.7.** The following holds:

(i) \( A_i \in FLO_{Kn,p}(\mathcal{K}) \Rightarrow Kn_i(L(\mathcal{K})) \subseteq L(\mathcal{A}_{x_i}) \). If \( Kn_i(x) \) is inwardly monotone in \( L(\mathcal{A}_{x_i}) \) then above \( \Leftrightarrow \) holds.

(ii) \( A_i \in GLO_{Kn,p}(\mathcal{K}) \Leftrightarrow [(\forall \Gamma/\varphi \in Ad(\mathcal{L}(\mathcal{K}))) (Kn_i(\Gamma) \vdash_{\mathcal{A}_x} Kn_i(\varphi) \text{ and } Kn_i(L(\mathcal{K})) \subseteq L(\mathcal{A}_{x_i}))]. \)

**Proof.** (i): Let \( A_i \in FLO_{Kn,p}(\mathcal{K}) \). Assume that \( Kn_i(\varphi) \in Kn_i(L(\mathcal{K})) \). Then \( \emptyset \Rightarrow \square_{s,ext} \varphi \) holds and, because of \( A_i \in FLO_{Kn,p}(\mathcal{K}) \) we may derive \( \vdash_{\mathcal{A}_x} Kn_i(\varphi) \), and thus obtain \( Kn_i(\varphi) \in L(\mathcal{A}_{x_i}) \). To verify the converse, when \( Kn_i(x) \) is inwardly monotone in \( L(\mathcal{A}_{x_i}) \), assume that \( Kn_i(L(\mathcal{K})) \subseteq L(\mathcal{A}_{x_i}) \). Suppose that for a set of formulas \( \Gamma \) and a formula \( \varphi \), \( [\Gamma \Rightarrow \square\varphi] \) holds. Then for any model \( \mathcal{M} \in \mathcal{K} \) and any element \( a \) of \( A_i \), \( (\mathcal{M}, a) \models -V \wedge [\Gamma] \Rightarrow (\mathcal{M}, a) \models -V \varphi \). Therefore, \( \wedge \Gamma \Rightarrow \varphi \in L(\mathcal{K}) \). From our assumption we have \( Kn_i(L(\mathcal{K})) \subseteq L(\mathcal{A}_{x_i}) \). Therefore we must conclude:

\[
Kn_i(\wedge \Gamma \Rightarrow \varphi) \in L(\mathcal{A}_{x_i}).
\]

If we admit the formula \( Kn_i(\wedge \Gamma) \) as hypothesis, from the enclosing above and using inward monotonicity of \( Kn_i(x) \) in \( L(\mathcal{A}_{x_i}) \) we immediately derive \( Kn_i(\varphi) \), thus \( Kn_i(\wedge \Gamma \Rightarrow_{\mathcal{A}_x} Kn_i(\varphi) \text{ and } A_i \in FLO_{Kn,p}(\mathcal{K}) \) holds.

(ii): First we assume that the right part of (ii) holds. Let \( [\Gamma \Rightarrow \square_{s,ext} \varphi] \).

By definition of \( \Rightarrow_{s,ext} \) for any substitution \( s \) of formulas in place of variables, \( \mathcal{K} \models \Gamma \) implies \( \mathcal{K} \models \varphi^{s} \), thus we conclude immediately \( \Gamma/\varphi \in Ad(\mathcal{K}) \).

Using subsequently the right part of (ii) we derive that \( Kn_i(\wedge \Gamma \Rightarrow_{\mathcal{A}_x} Kn_i(\varphi) \text{ and } \mathcal{A}_i \in GLO_{Kn,p}(\mathcal{K}) \). So we verified that \( \mathcal{A}_{x_i} \in GLO_{Kn,p}(\mathcal{K}) \).

Now we assume that \( \mathcal{A}_{x_i} \in GLO_{Kn,p}(\mathcal{K}) \). Suppose that \( \Gamma/\varphi \in Ad(\mathcal{K}) \), then \( L(\mathcal{K}) \) is closed w.r.t. the rule \( \Gamma/\varphi \) under using arbitrary substitutions \( s \), and it immediately implies \( \Gamma \Rightarrow_{s,ext} \varphi \) and using \( \mathcal{A}_{x_i} \in GLO_{Kn,p}(\mathcal{K}) \) we derive \( Kn_i(\wedge \Gamma \Rightarrow_{\mathcal{A}_x} Kn_i(\varphi) \text{ and } \mathcal{A}_{x_i} \in GLO_{Kn,p}(\mathcal{K}) \) follows \( \vdash_{\mathcal{A}_x} Kn_i(\varphi) \), i.e. we proved \( Kn_i(\varphi) \in L(\mathcal{A}_{x_i}) \). ■

In the right part of (ii) the second assertion follows from the first one but we want to point up the mentioned inclusion. Again this lemma gives
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us the necessary and sufficient condition how to avoid logical omniscience in the case of encoding knowledge as formulas \( K_n(x) \). And we are free to encode by \( K_n(x) \) everything what we want to say within language about \( x \). Assume \( K_n(x) := \neg K_i \neg x \), where \( K_i \) is \( i \)-th knows operation, then \( K_n(x) \), says that knowledge about \( x \) is possible.

We aim to explore to which extent the viewpoint on what an agent knows and what is logical omniscience, which we have considered for propositional logics, can be employed for first-order logics.

Let \( L \) be a language of a first order logic in a fixed signature. We assume that \( L \) has certain predicate symbols (functional ones can be also defined through, or to be presented themselves, constants are also allowed), and a countable infinite set of variables \( x_n, n \in \mathbb{N} \), \( y_n, n \in \mathbb{N} \), etc.

First, we have to define the meaning of a substitution \( s \) of formulas in the language of \( L \) in a set of formulas \( X \) (or merely a formula, \( X := \{ \theta \} \)) basing on \( L \).

Let \( P_j(t_{j1}, ..., t_{jn_j}) \), \( 1 \leq j \leq m \) be all predicate letters having occurrences in formulas \( X \) with terms \( t_{uk} \) shown above. We subconsciously associate these terms with variables they replace, therefore the order of the above terms is fixed and we keep it in mind.

Let \( \varphi_j(x_{j1}, ..., x_{jn_j}) \), \( 1 \leq j \leq m \) be a finite sequence of formulas in \( L \), each of which has at least free variables \( x_{j1}, ..., x_{jn_j} \) which are fixed and maybe still some additional variables (though we can also admit fictitious occurrences of any of these variables the order and pointed division are fixed). The result of acting \( X^s \) of the substitution \( s \) on \( X \) will be \( X^s := \{ \beta_{(\varphi_j)} \mid \beta \in X \} \), where any formula \( \beta_{(\varphi_j)} \) is the result of the following transformation:

Step (i): we exchange simultaneously and homogenously any free variable \( y_k \) in any formula \( \varphi_j(x_{j1}, ..., x_{jnj}) \) if \( y_k \) differs from \( x_{j1}, ..., x_{jnj} \) by a new and unique variable \( z_k \) which had no occurrences at all in all formulas of \( X \) and all formulas \( \varphi_j(x_{j1}, ..., x_{jnj}) \). As a result we obtain formulas \( \theta_j(x_{j1}, ..., x_{jnj}) \).

Step (ii): We replace in any formula \( \theta_j(x_{j1}, ..., x_{jnj}) \) all free occurrences of variables \( x_{jk} \) by terms \( t_{jk} \) and obtain formulas \( \gamma_j(t_{j1}, ..., t_{jnj}) \).

Step (iii): We replace in all formulas from \( X \) any occurrence of formulas \( P_j(t_{j1}, ..., t_{jnj}) \) by formulas \( \gamma_j(t_{j1}, ..., t_{jnj}) \), respectively.

The obtained set \( X^s \) is the result of the substitution \( s \) in \( X \). We call such substitutions permissible for \( X \).
We have taken so much precautions in this definition in order to avoid any collision between free and bounded occurrences of variables. This subtlety is really important, since it is a cornerstone of algebraic logic base how to define substitutions. Say, in general case, schema logics of almost all first order theories are undecidable (cf. [9]). Now we are in a position to describe strong external consequence for the first-order models.

**Definition 3.8.** For a finite set of first order formulas $\Gamma$ and a formula $\varphi$, we say $\varphi$ is a strong external consequence of $\Gamma$ on a class of first order models $\mathcal{K}$, (notation $\Gamma \Rightarrow_{\mathcal{K}}^{s.e.xt} \varphi$) iff for any substitution $s$ permissible for $\Gamma$ and $\varphi$, $[\mathcal{K} \models \Gamma^s] \Rightarrow [\mathcal{K} \models \varphi^s]$.

We turn to define knowledge formulas or information properties formulas, for agents with axiomatic systems in the first-order language (extending our approach to propositional logics).

Let $\mathcal{F}_{k,i}(P_1, \ldots, P_n, Q)$ be a fixed formula in $\mathcal{L}$ containing only predicate relations $P_1, \ldots, P_n$ and $Q$,

(i) $Q$ obligatory has real occurrences in this formula, $P_1, \ldots, P_n$ are optional, in particular, any of them can have fiction occurrences only,

(ii) dimensions of all mentioned predicates are arbitrary but fixed. No more restriction required, so we could admit equalities and inequalities between terms inside this formula, etc. We conventionally call it a formula expressing the knowledge of $i$-th agent.

Also we choose an axiomatic system $P\mathcal{Ax}_i$ for reasoning about knowledge of an agent $i$ in the given language $\mathcal{L}$.

For any substitution $s$ permissible for $\mathcal{F}_{k,i}$ which replaces letters $P_1, \ldots, P_n$ and $Q$ by first-order formulas $\varphi_1, \ldots, \varphi_n$ and $\varepsilon$, respectively, we agree to understand informally $\mathcal{F}_{k,i}^s$ as the statement that the formula $\varepsilon$ must be a consequence of $\varphi_1, \ldots, \varphi_n$ from the view-point of $i$-th agent. And formally,

**Definition 3.9.** If $P\mathcal{Ax}_i \vdash \mathcal{F}_{k,i}^s$ then we say that $i$-th agent knows that $\varepsilon$ is a consequence of $\varphi_1, \ldots, \varphi_n$, or that if $i$ knows all $\varphi_1, \ldots, \varphi_n$ then $i$ knows $\varepsilon$.

We could employ all $P_j$ to be tautologies, $\forall x(x = x)$, we will keep notation $\top$ for tautologies, then $P\mathcal{Ax}_i \vdash \mathcal{F}_{k,i}^s$ would mean $i$ simply knows $\varepsilon$. In this way, for example, we can say that $i$ knows that a formula $\varepsilon(x_1, x_2)$ always defines binary predicate which is well ordering, etc. Or, to say that
i knows that if some formulas define a model of Peano arithmetic then he knows that within this definition any two numbers have the greatest common divisor, etc. The above predicate letters $P_j$ are called premises, and $Q$ is the main predicate of $F_{k,i}$.

**Definition 3.10.** Let $F_{k,i}$ be the knowledge formula of $i$-th agent.

- Agent $A_i$ is fully logically omniscient about a class of first-order models $K$ (notation is $A_i \in \text{FLO}(K)$) iff, for any set of formulas $\alpha_1, ..., \alpha_n$ and any formula $\varphi$ and any permissible substitution $s$ for $F_{k,i}$, where $P_1 \mapsto_s \alpha_1, ..., P_n \mapsto_s \alpha_n$ and $Q \mapsto_s \varphi$,

$$[\alpha_1, ..., \alpha_n \Rightarrow^K_{\text{int}} \varphi] \Rightarrow P.Ax_i \vdash F_{k,i}^s.$$

- An agent $A_i$ is globally logically omniscient about a class of first-order models $K$ (notation is $A_i \in \text{GLO}(K)$) iff, for any set of formulas and permissible substitution $s$ as above,

$$[\alpha_1, ..., \alpha_n \Rightarrow^K_{\text{ext}} \varphi] \Rightarrow P.Ax_i \vdash F_{k,i}^s.$$

For any $F_{k,i}$, and any $P.Ax_i$, we fix the following notation:

$$F_{k,i}^{P.Ax_i} := \{ \varepsilon \mid \forall s, s \in \text{PerSub},$$

$$P_1 \mapsto_s \forall x(x = x), ..., P_n \mapsto_s \forall x(x = x), Q \mapsto_s \varepsilon, P.Ax_i \vdash F_{k,i}^s \} ,$$

i.e. it is the whole set of formulas which $i$ knows. We say that the knowledge of $A_i$ is weakly inwardly monotone ($F_{k,i}$ is w.i.m. for short) if

$$[P.Ax_i \vdash (F_{k,i})_{P_j,Q}^{\alpha_j \beta_{\alpha_j \varepsilon}}] \Rightarrow [P.Ax_i \vdash (F_{k,i})_{\alpha_j \varepsilon}].$$

What we are still missing is the definition of sequents $\Gamma/\theta$ admissible for first-order theories $Th(K)$. Let $\Gamma := \varepsilon_1, ..., \varepsilon_k$, and $\theta$ be arbitrary formulas of full predicate calculus (all predicate symbols are in the language but functions and constants are only from the language of $Th(K)$). We say $\Gamma/\theta$ is admissible for $Th(K)$ iff for any substitution $s$ permissible for $\Gamma, \theta$, assertion $\Gamma^s \in Th(K)$ implies $\theta^s \in Th(K)$. For any formulas $\alpha_1, ..., \alpha_n, \beta$, where $n$ is the number of premises of $F_{k,i}$, the notation $F_{k,i}, \alpha_1, ..., \alpha_n \mapsto_1 \beta$ is
an abbreviation for: $PAx_i \vdash (F_{k,i})^{(P_j,Q)}_{(\alpha_i,\beta)}$ and the mentioned substitution is permissible.

**Theorem 3.11.** The following holds:

(i) $A_i \in FLO_F(K) \Rightarrow Th(K) \subseteq F_{i,k,i}^{PAx_i}$. If $F_{k,i}$ is w.i.m. in $Ax_i$, then above $\Leftrightarrow$ holds.

(ii) $A_i \in GLO_F(K) \Leftrightarrow$, for any rule $\Gamma/\varphi$ with the number of premises as the number of ones in $F_{k,i}$, $[\Gamma/\varphi \in Ad(Th(K))] \Rightarrow F_{k,i}, \Gamma \hookrightarrow \beta$.

**Proof.** (i): Let $A_i \in FLO_F(K)$. Assume that $\varphi$ is a formula with at least $n$ free variables (fictitious occurrences included), where $n$ is the size of the main predicate of $F_{k,i}$. Assume $\varphi \in Th(K)$. Then it follows that $\emptyset \Rightarrow K_{int,} \varphi$. Then, in particular, the following $\forall x(x = x), ..., \forall x(x = x)(n - \text{times}) \Rightarrow K_{int,} \varphi$ holds. Since $A_i \in FLO_F(K)$ then for the permissible substitution $s$ substituting $T_1 := \forall x(x = x)$ for premises of $F_{k,i}$ and $\varphi$ for the main predicate of $F_{k,i}$: $P_1 \mapsto s_1, ..., P_n \mapsto s_n, Q \mapsto s_\varphi, PAx_i \vdash F_{k,i}^s$ holds. Thus, we conclude, that $\varphi \in F_{i,k,i}^{PAx_i}$ and $Th(K) \subseteq F_{i,k,i}^{PAx_i}$ which is what we needed.

To show the converse when $F_{k,i}$ is w.i.m. in $Ax_i$, assume that $Th(K) \subseteq F_{i,k,i}^{PAx_i}$. Suppose that for a set of formulas $\alpha_1, ..., \alpha_n$ and a formula $\varphi$ in $L$, where the substitution $P_1 \mapsto s_1, ..., P_n \mapsto s_n, Q \mapsto s_\varphi$, is permissible for $F_{k,i}$, $[\alpha_1, ..., \alpha_n \Rightarrow \varphi]$ holds. Then it follows, for any model $M \in K, [M \models \bigwedge_{1 \leq j \leq n} \alpha_j] \Rightarrow [M \models \varphi]$.

Therefore, $\bigwedge_{1 \leq j \leq n} \alpha_j \Rightarrow \varphi \in Th(K)$. From our above assumption we conclude: $Th(K) \subseteq F_{i,k,i}^{PAx_i}$. Consequently,

$PAx_i \vdash (F_{k,i})^{P_j,Q}_{(\alpha_i,\beta)}$

Using this derivation and w.i.m. of $F_{k,i}$ in $Ax_i$ we derive

$PAx_i \vdash (F_{k,i})^{P_j,Q}_{(\alpha_i,\beta)}$.

Thus $A_i \in FLO_F(K)$ holds.

(ii): Assume $Ax_i \in GLO_F(K)$. Suppose $\alpha_1, ..., \alpha_n/\beta \in Ad(Th(K))$ (where $n$ is the size of the quasi-premise of $F_{k,i}$). Then, for any sub-
stitution $s$ permissible for all $\alpha_j$ and $\beta$, $\forall j \alpha_j^s \in Th(K)$ implies $\beta^s \in Th(K)$. This entails $\alpha_1, ..., \alpha_n \models _{ext}^s \beta$. And using $Ax_i \in GLO_P(K)$ we conclude $P Ax_i \vdash (F_i)^{(P_j, Q)}$, so $i$ knows that $\beta$ is a consequence of all $\alpha_j$, i.e. $F_{k,i}, \alpha_1, ..., \alpha_n \leftarrow_i \beta$. Conversely, suppose that for any rule $\Gamma/\beta$ with the same number of premises as in $F_{k,i}$ and admissible for $Th(K)$, $F_{k,i}, \Gamma \leftarrow_i \beta$ holds. Let $\alpha_1, ..., \alpha_n \models _{ext}^s \beta$, and let $n$ be the number of premises in $F_{k,i}$. Then, for any permissible substitution $s$, $\forall j, \alpha_j^s \in Th(K)$ $\Rightarrow \beta^s \in Th(K)$, i.e. the sequent $\alpha_1, ..., \alpha_n / \beta$ is from $Ad(Th(K))$ and by our assumption we are in a position to derive $F_{k,i}, \alpha_1, ..., \alpha_n \leftarrow_i \beta$. Thus, as soon as $\alpha_1, ..., \alpha_n \models _{ext}^s \beta$, occurs, $F_{k,i}, \alpha_1, ..., \alpha_n \leftarrow_i \beta$ holds. Thus $Ax_i \in GLO_P(K)$. 

In the light of this theorem, a way to break logical omniscience in the last approach is to make again our agent to be not totally competent, i.e. to choose his axiomatic system which is weaker than the theory of model class under consideration, or to choose his knowledge formula not to be weakly inwardly monotone. Or still, for global omniscience, to take knowledge formula which does not embrace admissible rules.

References


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