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DECIDABILITY OF MODAL LOGICS $S_4 \oplus \alpha_N, S_4 \oplus \xi_{N+1}$
W.R.T. ADMISSIBLE INFERENCE RULES

Abstract

The paper examines hypotheses concerning decidability w.r.t. admissibility for some special modal logics (see [1]). We prove decidability w.r.t. admissible rules for the logics $S_4 \oplus \xi_{N+1}$, generated by all finite, rooted, reflexive and transitive frames, maximal clusters of which have at most $N$ elements. Moreover there is a proof given for decidability w.r.t. admissible rules for modal logics $S_4 \oplus \alpha_N$. Their characteristic classes are finite, rooted, reflexive, transitive frames, which have at most $N$ maximal clusters.

The study of admissible inference rules plays an important role in modern investigations of non-standard logics. Traditionally, this research has many aspects, such as exploration of decidability w.r.t. admissible rules, finding bases of admissible rules, solutions for logical equations with metavariables, validity of quasi-identities on free algebras of quasi-varieties, etc. In [4], an approach to investigation of inheritance of admissible inference rules was initiated. Additionally, the study of strong decidability (see [1]) of general properties of formal calculi leads to answer some particular questions concerning admissible rules.

L. L. Maksimova [1] formulated conjectures concerning decidability w.r.t. admissibility for certain special logics. Their confirmation would give us strong decidability of interpolation property for certain modal logics over $S_4$ and, moreover, some other properties. In this paper we prove decidability of logics $S_4 \oplus \xi_{N+1}$ w.r.t. admissible rules, which solves the open question given in [1]. We also show decidability of modal logics $S_4 \oplus$
\(\alpha_N\) w.r.t. admissible rules. The verification decidability of logics \(S4 \oplus \xi_{N+1}\) w.r.t. admissible rules is rather simple relatively - it is sufficient to apply only the property of branching below \(m\) and the effective \(m\)-drop points property, and then to use the algorithmic criterion of decidability for admissible inference rules from [2,3]. For modal logics such as \(S4 \oplus \alpha_N\), the solution of this question is not possible, because these logics fail to have the property of branching below \(m\). So we need some revision of previous methods to solve this problem (cf. Theorem 9 and 12).

Basic definitions, notation, theorems and construction, concerning inference rules could be found, for example, in [2,3]. We only recall here some necessary theorems and constructions.

**Theorem 1.** [2] Let \(K_n = \langle W_n, R_n, V_n \rangle, n \in \mathbb{N}\) be a sequence of \(n\)-characterizing models for the modal logic \(\lambda\). The inference rule \(r := \alpha_1(\bar{x}), \ldots, \alpha_n(\bar{x})/\beta(\bar{x})\) is admissible in \(\lambda\) iff for every \(n \in \mathbb{N}\) and each definable valuation \(S\) of variables form \(r\) in \(K_n\) the following holds. If \(S(\alpha_n) = W_n, \ldots, S(\alpha_1) = W_n\) then \(S(\beta) = W_n\), as well (that is, if \(r\) is valid in \(K_n\) with respect to all definable valuations).

**Theorem 2.** [2] Any element of the model \(Ch_{K_4}(n)\) is definable. In particular, for any normal modal logic \(\lambda\), extending \(K_4\), and having the fmp, every element of the model \(Ch_{\lambda}(k)\) is definable.

**Lemma 3.** [2] Let \(\lambda\) be a modal logic extending \(K_4\) or a superintuitionistic logic which have the fmp. If there is a valuation \(S\) of the variables of inference rule \(r := \alpha_1(\bar{x}), \ldots, \alpha_n(\bar{x})/\beta(\bar{x})\) in the frame of some \(n\)-characterizing model \(Ch_{\lambda}(n)\), which invalidates \(r\), then there is a valuation \(S\) of these variables in \(Ch_{\lambda}(k)\), where \(k\) is a number of variables in \(r\), which also disproves \(r\).

Let a model \(M = \langle M, R, V \rangle\) is based on a rooted, reflexive, transitive frame, which does not have infinite ascending chains of clusters. Let \(Y\) be a finite set of formulae, closed w.r.t. subformulae, all propositional letters of formulae \(Y\) are included in \(\text{Dom}(V)\) and \(X \subseteq Y\). Notation \(S_m(W)\) reads the submodel of \(W\) which consists of all clusters from \(W\), with depth at most \(m\); and \(S_m(W)\) (\(m\)-slice of \(W\)) are all clusters of \(W\) with depth \(m\).

We define the set of quasi-maximal elements of \(A \subseteq W\) as follows:

\[qm(A) = \{a | a \in A, \forall x \in W (x \in a^{R<} \Rightarrow x \notin A)\}\]

Let \(V_\alpha(X) = \{a | a \in W, \forall \alpha \in X, \forall \beta \in (Y - X) (a \models_\alpha \alpha \land (a \not\models_\beta)\}\) and \(X_{m,W} = \{a | a \in (W - \ldots\}

$S_m(W), \ a \in qm(V_4(X)))$. The submodel $\text{max}(Y,W,m)$ consists of all clusters from $W$, which includes some elements of sets $X_{m,W}$ for all subsets $X \subseteq Y$. The submodel $\text{max}(Y,W,m) \cup S_m(W)$ of the model $W$ consists of all clusters $\text{max}(Y,W,m)$ and $S_m(W)$ with the accessibility relation between them and the valuation transferred from the original model $W$.

We will also use the following abbreviations: let $x$ be an element of the model $M = \langle W, R, V \rangle$, $Y$ be a set of formulae, all propositional letters of formulae $Y$ are included in $\text{Dom}(V)$. Then: $S(x,Y) = \{ \alpha | \alpha \in Y, x \parallel_{V_\alpha} \alpha \}$, $\Diamond(x,Y) = \{ \alpha | \alpha \in Y, x \parallel_{V_\alpha} \Diamond \alpha \}$. It is clear, that on reflexive models the following inclusion: $S(x,Y) \subseteq \Diamond(x,Y)$ holds.

For the proof of decidability of modal logics w.r.t. admissibility we will use the algorithmic criterion for admissibility, expounded, for example, in monograph [2]:

**Theorem 4.**[2] *(The algorithmic criterion for admissibility)* Let $\lambda$ be a modal logic, having the fmp, extending $K4$ and such that:

a) $\lambda$ has the property of branching below $m$,

b) $\lambda$ has the effective $m$-drop points property,

c) there is an algorithm which determines whether a given finite frame is a $\lambda$-frame (which holds if, in particular, $\lambda$ is finitely axiomatizable).

Then there is an algorithm which determines admissibility of inference rules in $\lambda$.

**Corollary 5.**[2] For any $N \geq 1$ the modal logics $K4 \oplus \alpha_N$, $K4 \oplus \xi_{N+1}$, $S4 \oplus \alpha_N$, $S4 \oplus \xi_{N+1}$ have the fmp and are decidable.

For the verification of decidability w.r.t. admissibility of modal logics $S4 \oplus \alpha_N$ at first we will show that:

**Lemma 6.** For any $N \geq 1$ the modal logic $= S4 \oplus \alpha_N$ has the effective $1$-drop points property.

**Proof:** Let $\mathcal{M} = \text{max}(Y,W,1) \cup S_1(W)$. Then the depth of $= \text{max}(Y,W,1) \cup S_1(W)$ can be bounded by $2^v + 1$, where $v$ is a number of formulae in the set $Y$. As a condensing $p$-morphism $f$ we will take the procedure "contraction" for finite models whose depths are bigger than 1. We remind that the procedure of condensing for clusters of $d$-slice is as follows:

a) In each cluster $C$ of $d$-slice we contract elements $x, y \in C$, if $x$ and $y$ have the same valuation of all propositional variables;
b) We identify clusters $C_1$ and $C_2$ from d-slice, if $C_1^{R<} = C_2^{R<}$ and $C_1$ and $C_2$ are isomorphic as models.

The model $M_1 = f(M)$ have the following properties: $S_1(M_1) = f(S_1(W)) = S_1(W)$, the model $M_1$ is also based on the $\lambda$-frame, depth of the model $M_1$ is not more than $2^n + 1$ and the rule r will be also disproved on the model $M_1$. Now we construct the function, estimating the number of elements of condensed model $M_1$. We will introduce the recurrent function $m(d, u, 2^n)$, conducting estimation of a clusters’ number in a condensed open rooted submodel $F_k'(d, u, 2^n) \subseteq M_1 \setminus S_1(W)$ with the depth $d$ and the recurrent function $n(d, u, 2^n)$, estimating the number of the different rooted submodels $F_k'(d, u, 2^n) \cup S_1(W) \subseteq M_1$.

It is evident that $m(1, u, 2^n) = 1$ and $n(1, u, 2^n) = (2^{2^n} - 1) \cdot (2^u - 1)$. Consider the arbitrary submodel $F_k'(s, u, 2^n) \cup S_1(W) \subseteq M_1$. Clearly, we can take estimations: $m(s, u, 2^n) = 1 + \sum_{j=1}^{s-1} m(j, u, 2^n) \cdot n(j, u, 2^n)$ and $n(s, u, 2^n) = \sum_{h=s}^{m(s, u, 2^n)} ((2^{u+h-1} - 1) \cdot (2^{2^n} - 1))^h$. After all, the number of elements of the model $M_1$ can be bounded as follows:

$$|M_1| \leq u + 2^n \cdot \left( \sum_{k=1}^{2^n} m(k, u, 2^n) \cdot n(k, u, 2^n) \right) = g(u, v). \quad (1)$$

Let a model $M$ be based on a reflexive and transitive frame which does not have infinite ascending chains of clusters. We will also require that every cluster of the model $M$ has no two elements with the same valuation of propositional variables, and that there no two distinct clusters are isomorphic as submodels among maximal clusters of the model $M$. Let $\{p_1, ..., p_k\} = Dom(V)$ be propositional variables from the domain of the model $M$. Let us consider the class of formulae $RCM(M) := \{\varphi(C_1, ..., C_f) | C_k \in S_1(M)\}$ where:

$$\varphi(C_1, ..., C_f) = \left( \bigwedge_{C_i \in \{C_1, ..., C_f\}} \rho(C_i) \right) \land \left( \bigwedge_{C_j \notin \{C_1, ..., C_f\}} \neg \rho(C_j) \right)$$

$$\rho(C) = \Box \Box \left( \bigwedge_{a \in C} \Box \Box \gamma(a) \right) \land \left( \bigwedge_{b \notin C, \forall a \in C \gamma(b) \neq \gamma(a)} \neg \Box \gamma(b) \right)$$
Lemma 7. Suppose there is a valuation $S$ of $k$ variables of the given inference rule $r := \alpha_1(x), \ldots, \alpha_n(x) /\beta(x)$ on the frame of $n$-characterizing model $\text{Ch}_S(\alpha_N, \alpha_N(k))$ for some modal logic $S_4 \oplus \alpha_N$. Suppose $S$ invalidates the inference rule $r$. Then there is a $(S_4 \oplus \alpha_N)$-model $M = \{W, R, S\} = \bigcup \mathcal{M}_j$, where the domain of $S$ includes all propositional variables from $r$, such that:

a) the frame $S_1(M)$ is isomorphic to the frame $S_1(\text{Ch}_S(\alpha_N, \alpha_N(k)))$;

b) the number of elements $M$ is bounded by: $|M| \leq g(l, q) \cdot (2^{2^n} - 1)$, where $l = S_1(\text{Ch}_M(k)), q = |\text{Sub}(r)|$, $\text{Sub}(r)$ is the set of all subformulae of the inference rule $r$, with their negations, but $-\beta \notin \text{Sub}(r)$, and $g(l, q)$ is the function, defined in (1);

c) the model $M$ invalidates the rule $r$;

d) for any anti-chain $E$ of clusters from the model $\mathcal{M}_j$ and for $Y = \text{Sub}(r) \cup RCM(M)$, such that the frame $[E]^{R \subseteq}$ has not more than $N$ maximal clusters, there is an element $X_E$ in the model $\mathcal{M}_j$ such that: $\diamond (X_E, Y) = \{\diamond (y, Y) | y \in E\} \cup S(X_E, Y)$;

e) each model $\mathcal{M}_j$ is an open submodel of the model $\text{Ch}_S(\alpha_N, \alpha_N(k))$.

Proof: According to Lemma 3, the inference rule $r$ is invalidated under some valuation $S_2$ on the frame of the $k$-characterizing model $\text{Ch}_S(\alpha_N, \alpha_N(k)) = \bigcup \mathcal{H}_j$, where $k$ is the number of variables in $r$. For each subset of formulae $Y \subseteq \text{Sub}(r)$ and each formula $\varphi \in RCM(\text{Ch}_S(\alpha_N, \alpha_N(k)))$ we take the set $Z = Y \cup \{\varphi\}$ and choose in $\mathcal{H}_j$ some element $X_Z$ (if there is such element in model $\mathcal{H}_j$), such, that $S(X_Z, \text{Sub}(r) \cup RCM(\text{Ch}_S(\alpha_N, \alpha_N(k)))) = Z$.

We construct the submodel $\mathcal{K}_Z$ as an open submodel of $\mathcal{H}_j$, generated by the element $X_Z$. Let the submodel $\mathcal{K}_j = \bigcup \mathcal{K}_Z$ be an open submodel of $\mathcal{H}_j$, generated by all such elements $X_Z$. It is not difficult to verify that the model $\bigcup \mathcal{K}_j$ will be also based on $(S_4 \oplus \alpha_N)$-frame and that the rule $r$ is also invalidated on it. The number of all possible subsets $Z$ does not exceed $2^{2^{\text{Sub}(r)}} + |RCM(\text{Ch}_S(\alpha_N, \alpha_N(k))))| = 2^{2^{\text{Sub}(r)}} + (2^{2^n} - 1)$, and the
number of components $H_j$ is not more than $2^{2^k}$, hence the model $\prod K_j$ will be also finite. By Lemma 6, for each component $K_j$, containing the set $\text{max}(\text{Sub}(r) \cup CHS_{4\oplus N}(k))$, there is condensing $p$-morphism $f_j$ from $K_j$ onto a finite $\lambda$-model $M_j$. Let $M = (W, R, V) = \prod M_j$, the model $M$ obviously satisfies properties (a), (b) and (e) from the formulation of this lemma. Property (c) also takes place because rule $r$ will be disproved on the model $\prod K_j$.

It is possible, though not so simple, to show that the property (d) holds. However, the proof must be omitted here, for the sake of brevity.

**Lemma 8.** Suppose that there exists a $\lambda$-model for some modal logic $S_{4\oplus N}$ $M = (W, R, S) = \prod M_j$ which satisfies properties (a) - (e) from Lemma 7. Then there is a definable valuation $S$ for $k$ variables of the inference rule $r$ in the frame of the $k$-characterizing model $CHS_{4\oplus N}(k)$, such that $S$ also invalidates the rule $r$ in the frame of $CHS_{4\oplus N}(k)$.

**Proof:** Consider an $n$-characterizing model $CHS_{4\oplus N}(k) = \prod H_j$. According to the properties (a) and (e) from Lemma 7 each model $M_j$ is an open submodel of $H_j$ and frames, generated by submodels $S_1(H_j)$ and $S_1(M_j)$, are isomorphic. Hence, we can assume that all elements of the model $M$ are elements of the model $CHS_{4\oplus N}(k)$. According to Theorem 2 each element of this model is definable, consequently we can construct the valuation $V$ for $k$ variables of the rule $r$ (according to the property (e) from Lemma 7) on the frame of the model $M$ as an open subframe of the frame of the model $CHS_{4\oplus N}(k)$ which will coincide with the valuation $S$ of the model $M$. Then we will abbreviate $M^{ch} = (W, R, V) = \prod M^{ch}_j$, supposing that $|M^{ch}| \subseteq |CHS_{4\oplus N}(k)|$.

We will extend the valuation $V$ onto whole the frame of model $CHS_{4\oplus N}(k)$: we will construct a sequence of subsets $\sum(x, t)$ on every $H_j$ for $\forall x \in |CHS_{4\oplus N}(k)|$, $\forall t \leq m_1$, where $m_1$ is a quantity of elements in the model $M^{ch}_j$.

The sequence will satisfy the following properties:

(a) $\forall x \in |CHS_{4\oplus N}(k)|$, $\sum(x, t) \subseteq \sum(x, t + 1)$; $\forall x_1, x_k \in |CHS_{4\oplus N}(k)|$; $x_i \neq x_k$; $\sum(x_i, t) \cap \sum(x_k, t) = \emptyset$;

(b) each subset $\sum(x, t)$ is definable under the valuation $V$ in the model $H_j$; $\forall x \in |CHS_{4\oplus N}(k)|$, $\forall t \leq m_1$, $\exists \alpha(x, t)$: $\forall y \in |H_j|$ $y \models \alpha(x, t)$ $\iff$ $y \in \sum(x, t)$;
(c1) \( \forall t \ 0 \leq t \leq m_1, \bigcup \{ \{ x, t \} | x \in |M^{ch}_j | \} \) forms an open subframe of the frame of the model \( \mathcal{H}_j \);

(d1) if the valuation \( V \) for variables of the inference rule \( r \) on the frame \( \bigcup \{ \{ x, t \} | x \in |M^{ch}_j | \} \) satisfies \( \forall p \in Var(r), \ V(p) = \bigcup \{ \{ x, t \} | x \in |M^{ch}_j |, \ x \models_v p \}, \) that \( \forall \alpha \in \text{Sub}(r) \cup RCM(M_j), \ \forall x \in |M^{ch}_j |, \ \forall a \in \sum(x, t) a \models_v \alpha \leftrightarrow x \models_v \alpha ; \)

(e1) If \( \forall t \ 0 \leq t \leq m_1, \forall y \in |\mathcal{H}_j |, \ y \not\in \bigcup \{ \{ x, t \} | x \in |M^{ch}_j | \} \) holds, then there is a subset \( W \) : \( |W| = t + 1, \ W \subseteq |M^{ch}_j |, \) such that \( \forall x \in |W| \ yR^c \cap \sum(x, t) \neq \emptyset ; \)

We will describe a construction of a sequence of subsets \( \sum(x, t) \) by induction on \( t \). Let \( \forall x \in |M^{ch}_j | \sum(x, 0) = \{ x \} \). Assume for any \( \forall x \in |M^{ch}_j | \) subsets \( \sum(x, g) \) for all \( g \leq t \) are constructed. Take an arbitrary \( (t + 1) \)-elements subframe \( D \) of elements of the model \( M^{ch}_j \), where the subframe \( DR^c \subseteq \) has not more than \( N \) maximal clusters. Let \( \mathcal{E}_D \) be a set of \( = \) representatives of minimal clusters from \( DR^c \subseteq \) of the model \( \mathcal{H}_j \) (one representative from each cluster). Then obviously: \( \{ \{ \Diamond(y, \text{Sub}(r) \cup RCM(M_j)) | y \in \mathcal{E}_D \} = \{ \{ \Diamond(y, \text{Sub}(r) \cup RCM(M_j)) | y \in \mathcal{E}_D \} \cup \mathcal{S}(X, \text{Sub}(r) \cup RCM(M_j)) \). For the anti-chain \( \mathcal{E}_D \) we fix an element \( \chi_D \), satisfying the property (d) from the formulation of Lemma 7: \( \Diamond(X, \text{Sub}(r) \cup RCM(M_j)) = \bigcup \{ \Diamond(y, \text{Sub}(r) \cup RCM(M_j)) | y \in \mathcal{E}_D \} \cup \mathcal{S}(X, \text{Sub}(r) \cup RCM(M_j)) \). For this element \( \chi_D \) we can introduce formulae \( q(D) \) and \( \chi(D) \):

\[
q(D) = \bigwedge_{x \in D} \Diamond \alpha(x, t) \land \bigwedge_{x \not\in D, x \in |M^{ch}_j |} \neg \Diamond \alpha(x, t) \land \bigvee_{x \in |M^{ch}_j |} \alpha(x, t),
\]

\[
\chi(D) = q(D) \land \Box \bigwedge_{x \in |M^{ch}_j |} \neg \alpha(x, t) \rightarrow \Diamond q(D) \land \bigvee_{x \in D} \alpha(x, t) \lor q(D).
\]

where \( \alpha(x, t) \) are formulae for sets \( \sum(x, t) \), satisfying the property (b1). Let \( \sum(x, t + 1) = \sum(x, t) \cup \{ \{ V(\chi(D)) | X = x \} \} \). It can be shown, that \( \bigcup \{ \{ x, t \} | x_i \in |M^{ch}_j |, \ 0 \leq t \leq m_1 \} = \mathcal{H}_j \) in details.

By this procedure we extend the valuation \( V \) from the model \( M^{ch} \) onto whole the frame of the model \( Ch_{S4@\alpha_N}(k) \). It can be seen that this new definable valuation \( V \) disproves the rule \( r \) on the frame of the model \( Ch_{S4@\alpha_N}(k) \).
From Theorem 1 and Lemmas 7, 8 it follows that the necessary and sufficient condition of admissibility of the inference rule $r$ in the modal logic $S4 \oplus \alpha_N$ is valid on all $\lambda$-models $M$ of the logic $S4 \oplus \alpha_N$, satisfying the properties (a) - (e) from Lemma 7. Thus we proved:

**Theorem 9.** For any $N \geq 1$ the modal logic $S4 \oplus \alpha_N$ is decidable w.r.t. admissibility.

For the proof of decidability w.r.t. admissibility of modal logics $S4 \oplus \xi_{N+1}$ we will show that:

**Lemma 10.** For any $N \geq 1$ the modal logic $S4 \oplus \xi_{N+1}$ has the property of branching below 1.

**Proof:** Let us show that $S4 \oplus \xi_{N+1}$ admits reflexive $d$-branching below 1 for any number $d$. As a $(S4 \oplus \xi_{N+1})$-frame we can take the rooted, reflexive and transitive frame whose depth is 2, and which has exactly $d$ maximal one-element reflexive clusters. Adding the new reflexive root to arbitrary $(S4 \oplus \xi_{N+1})$-frame $W$ of finite depth with retaining condition of transitivity on the frame $W_1$ cannot increase a number of elements in maximal clusters, and the obtained frame $W_1$ holds as a $(S4 \oplus \xi_{N+1})$-frame. Consequently, the frame $W_1$, obtained from the finitely generated subframe of any component $W$ from $Ch_{S4 \oplus \xi_{N+1}}(n)$ by adding new one-element reflexive root, will also be $(S4 \oplus \xi_{N+1})$-frame. Q.E.D.

**Lemma 11.** For any $N \geq 1$ the modal logic $S4 \oplus \xi_{N+1}$ has the effective 1-drop points property.

The proof of this lemma is completely similar to the proof of Lemma 6 for the modal logic $S4 \oplus \alpha_N$.

Applying Lemmas 10, 11, and Corollary 5 to Theorem 4 (the algorithmic criterion of admissibility), we obtain:

**Theorem 12.** There is an algorithm, recognizing admissibility of inference rules in modal logics $S4 \oplus \xi_{N+1}$.

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References


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