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# RESTRICTING THE CONTRACTION AXIOM IN DUMMETT'S LC: A SUBLOGIC OF LC WITH THE CONVERSE ACKERMANN PROPERTY, THE LOGIC $LC^o$

**Abstract**

$LC^o$ with the Converse Ackermann Property is defined as the result of restricting Contraction in LC. Intuitionistic and Superintuitionistic Negation is shown to be compatible with the C.A.P..

*Keywords.* Superintuitionistic Logic, Contraction Axiom, Converse Ackermann Property

## 1. Introduction

In [3] it is shown how to define the logic CIr: Dummett's LC without the contraction but with the reductio axiom. In this paper we consider the possibility of extending CIr with the restricted contraction axiom

rW. $(A \to (A \to (B \to C))) \to (A \to (B \to C))$

that is, the contraction axiom

W. $(A \to (A \to B)) \to (A \to B)$

restricted to the case in which $B$ is an implicative formula ($A$ is implicative iff $A$ is of the form $B \to C$). We'll show:

1. rW is not a CIr theorem

2. CIr can not be extended with rW: the result would collapse in LC

3. CIr can be extended with rW provided assertion is also restricted to:

rA. $A \to ((A \to (B \to C)) \to (B \to C))$

that is, the assertion axiom

A. $A \to ((A \to B) \to B)$

restricted to $B$ an implicative formula.

We shall concentrate upon this last logic, namely LC with restricted assertion and restricted contraction. Let us refer to this logic by LC$^o$. The point in studying LC$^o$ is:

4. LC$^o$ is, so we think, an intersting sublogic of LC with the Converse Ackermann Property (C.A.P.).

C.A.P. is first defined in [1], where the problem as to which logics do possess C.A.P. is posed. A partial solution is offered in [2] concerning subintuitionistic logics. Defining LC$^o$ widens the scope of logics with the C.A.P. and, significantly, shows that intuitionistic and superintuitionistic negation are in fact C.A.P. compatible. Finally,

5. Properly adapted complete ternary relational semantics for LC$^o$ are offered.

## 2. The Logic CIr

The Logic CIr is motivated, defined and axiomatized in [3]. For the present purposes, it is convenient to work with an alternative axiomatization of both positive and negative fragments of CIr, containing the following axioms, rules and falsity constant F:

Axioms

A1. $A \to (B \to A)$

A2. $(A \to B) \to ((B \to C) \to (A \to C))$

A3. $A \to ((A \to B) \to B)$

A4. $(A \wedge B) \to A \qquad (A \wedge B) \to B$

A5. $((A \to B) \wedge (A \to C)) \to (A \to (B \wedge C))$

A6. $A \to (A \vee B) \qquad B \to (A \vee B)$

A7. $((A \to C) \wedge (B \to C)) \to ((A \vee B) \to C)$

A8. $(A \wedge (B \vee C)) \to ((A \wedge B) \vee C)$

A9. $(A \to B) \vee (B \to A)$

A10. $A \to ((A \to F) \to F)$

A11. $(A \to (A \to F)) \to (A \to F)$
A12. $F \to B$
Rules of Inference: Modus Ponens (if $\vdash A \to B$and $\vdash A$, then $\vdash B$) and Adjunction (if $\vdash A$ and $\vdash B$, then $\vdash A \wedge B$).

## 3. rW is not a theorem of CIr

Consider the following set of matrices, where the only designated value is 2 and F is assigned the value 1:

| $\to$ | 0 | 1 | 2 | $\vee$ | 0 | 1 | 2 | $\wedge$ | 0 | 1 | 2 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 2 | 2 | 2 | 0 | 0 | 1 | 2 | 0 | 0 | 0 | 0 |
| 1 | 1 | 2 | 2 | 1 | 1 | 1 | 2 | 1 | 0 | 1 | 1 |
| 2 | 0 | 1 | 2 | 2 | 2 | 2 | 2 | 2 | 0 | 1 | 2 |

This set verifies CIr but falsifies rW only when A=1, B=2, C=0 (Cfr. [4], §5, where the same result is rendered by a frame argument).

## 4. CIr plus rW

We note that contraction is derivable in CIr plus rW. As $((A \to A) \to B) \longleftrightarrow B$ is a theorem, replace in rW "$B$" by " $A \to A$" and "$C$" by "$B$".

## 5. The Logic LC$^o$

The Logic LC$^o$ is the result of deleting A3 and A12 in CIr and adding the axioms
A3'. $A \to ((A \to (B \to C)) \to (B \to C))$
A3". $(A \to (A \to (B \to C))) \to (A \to (B \to C))$
and
A12'. $F \to (A \to B)$
Thus, we note, LC$^o$ is the result of restricting in LC contraction, assertion and E Contradictione Quodlibet ($F \to B$) to the case in which B is an implicative formula.

# 6. Converse Ackermann Property

Consider the following set of matrices where F is assigned the value 0 and 2 is the only designated value

| $\to$ | 0 | 1 | 2 |
|---|---|---|---|
| 0 | 2 | 0 | 2 |
| 1 | 2 | 2 | 2 |
| 2 | 0 | 0 | 2 |

| $\vee$ | 0 | 1 | 2 |
|---|---|---|---|
| 0 | 0 | 0 | 2 |
| 1 | 0 | 1 | 2 |
| 2 | 2 | 2 | 2 |

| $\wedge$ | 0 | 1 | 2 |
|---|---|---|---|
| 0 | 0 | 1 | 0 |
| 1 | 1 | 1 | 1 |
| 2 | 0 | 1 | 2 |

This set verifies LC$^o$ but falsifies $(A \to (A \to B)) \to (A \to B)$ only when $A$=0, $B$=1 and $A \to ((A \to B) \to B)$ only when $A$=2, $B$=1. We show that LC$^o$ has the C.A.P.: let $(A \to B) \to C$ be a wff in which $C$ contains nor $\to$ neither $F$. Assign all variables in the value 1. Then,

$v((A \to B) \to C) = 0$.

We note that, for example, the following are LC $^o$ theorems (recall that $\neg A = A \to F$):

$(A \to \neg B) \to (B \to \neg A)$

$A \to \neg\neg A$

$(A \to \neg A) \to \neg A$

$(A \to B) \to ((A \to \neg B) \to \neg A)$

$\neg A \to (A \to \neg B)$

We remark that if $F \to B$ is introduced instead of A12', the resulting logic would not have the C.A.P.

# 7. Semantics

A LC$^o$ model is a quadruple $< K, S, R, \models>$ where $K$ is a set, $S$ a non-empty subset of $K$ and $R$ is a ternary relation on $K$ subject to the following definitions and postulates for all $a, b, c, d \in K$ with quantifiers ranging over $K$:

d1. $a \leq b =_{def} \exists x Rxab$

d2. $R^2abcd =_{def} \exists x(Rabx$ and $Rxcd)$

d3. $R^3abcde =_{def} \exists x\exists y(Rabx$ and $Rxcy$ and $Ryde)$

P1. $a \leq a$

P2. $(a \leq b$ and $Rbcd) \Rightarrow Racd$

P3. $R^2abcd \Rightarrow \exists x(Racx$ and $Rbxd)$

P4. $R^2abcd \Rightarrow R^3abbcd$
P5. $R^2abcd \Rightarrow R^2bacd$
P6. $Rabc \Rightarrow a \leq c$
P7. $(Rabc$ and $Rade) \Rightarrow (b \leq e$ or $d \leq c)$
P8. $(Rabc$ and $c \in S) \Rightarrow \exists x(x \in S$ and $Rbax)$
P9. $(Rabc$ and $c \in S) \Rightarrow \exists x \exists y(Rabx$ and $Rxby$ and $y \in S)$
P10. $a \notin S \Rightarrow (not - Rabc$ or $c \models A$ for any wff $A)$.

Finally, $\models$ is a valuation relation from $K$ to the sentences of LC$^o$ satisfying the following conditions for all $p, A, B$ and $a, b, c \in K$
(i) $(a \models p$ and $a \leq b) \Rightarrow b \models p$
(ii) $a \models A \vee B$ iff $a \models A$ or $a \models B$
(iii) $a \models A \wedge B$ iff $a \models A$ and $a \models B$
(iv) $\alpha \models A \rightarrow B$ iff for all $b, c \in K, (Rabc$ and $b \models A) \Rightarrow c \models B$
(v) $a \models F$ iff $a \notin S$

A is valid in LC$^o$ ($\models_{LC^o} A$) iff $a \models A$ for all $a \in K$ in all models. Proof of the semantic consistency of LC$^o$ [if $\vdash_{LC^o} A$, then $\models_{LC^o} A$] is left to the reader. (Cf. [2] for a general strategy. As for the validity of A10, A11 and A12' use, respectively, P8, P9 and P10).

## 8. Completeness of LC$^o$

We begin with some definitions. Let $K^T$ be the set of all theories (a theory is a set of formulas closed under adjunction and provable entailment). Let $R^T$ be defined as follows: for all formulas $A, B$ and $a, b, c, d \in K^T$ :

$R^Tabc \Leftrightarrow [(A \rightarrow B \in a$ and $A \in b) \Rightarrow B \in c]$

Further, let $K^C$ be the set of all prime non-null theories [a theory $a$ is prime if whenever $A \vee B \in a$, then $A \in a$ or $B \in a$; a theory is null iff no formula belongs to it]. And let $S^C$ be the set of all consistent theories [$a$ is consistent iff the negation of a theorem does not belong to $a$]. Finally, let $\models^C$be defined for any wff $A$ and $a \in K^C$ as follows: $a \models^C A \Leftrightarrow A \in a$.Then, the LC$^o$ canonical model is the quadruple $< K^C, S^C, R^C, \models^C >$.

We now prove completeness beginning with some previous lemmas. In Lemmata 1, 5 and 6 we refer to previous work in substructural logics: these lemmata are proved or else they are easily derived from the results there refered.

Lemma 1. *Let $A$ be a wff, $a$ a non-null element in $K^T$ and $A \notin a$. Then, $A \notin x$ for some $x \in K^C$ such that $a \subseteq x$.*

Proof. See [4].

Lemma 2. *For any $a \in K^T, F \in a$ iff $a$ is inconsistent.*

Proof. Easy using A11.

Lemma 3. *$S^C$ is not empty.*

Proof. As $\not\models_{LC^o} F$, we have $\not\vdash_{LC^o} F$ by semantic consistency. Then, Lemma 1 is applicable and there is some $x \in K^C$ such that $LC^o \subseteq x$ and $F \notin x$. Thus, $x$ is consistent. Therefore, $x \in S^C$.

Lemma 4. The canonical $\models^C$is a valuation relation satisfying conditions (i)-(v) [§6].

Proof. Clauses (i)-(iv) can be proved, for example, as in [3]. Clause (v) holds by Lemma 2.

Now, concerning the proofs of the following two Lemmas, that of Lemma 5 is immediate and the proof of Lemma 6 easily follows from [4]:

Lemma 5. *Let $a, b$ non-null theories. The set $x = \{B : \exists A(A \rightarrow B \in a$ and $A \in b)\}$ is a non-null theory such that $R^T abx$*

Lemma 6. *Let $R^T abc, a$ a non-null element in $K^T, b \in K^T$ and $c$ a prime member in $K^T$. Then, $R^T xbc$ for some $x$ in $K^C$ such that $a \subseteq x$.*

Lemma 7. *The canonical postulates hold in the $LC^o$ canonical model.*

Proof. See [2] and [3] for P1-P7. We now prove that P8, P9 and P10.

P8. $(R^C abc$ and $c \in S^C) \Rightarrow \exists x(x \in S^C$ and $R^C bax)$

Define (Cf. Lemma 5) the non-null theory $y = \{B : \exists A(A \rightarrow B \in b$ and $A \in a)\}$ such that $R^T bay$. We prove

the consistency of $y$ by reductio. If $F \in y$ (Lemma 2), then, by definition of $y$, $A \rightarrow F \in b$ and $A \in a$. By A10,

$(A \rightarrow F) \rightarrow F \in a$. Since $R^C abc$ is given, deduce $F \in c$ contradicting the hypothesis. Now apply Lemma 1 to

extend $y$ to some $x \in S^C$ such that $R^C bax$.

P9. $(R^C abc$ and $c \in S^C) \Rightarrow \exists x \exists y(R^C abx$ and $R^C xby$ and $y \in S^C)$

Define with Lemma 5 the non-null theories:
$u = \{B : \exists A(A \to B \in a \text{ and } A \in b)\}$
$w = \{B : \exists A(A \to B \in u \text{ and } A \in b)\}$
satisfying $R^T abu$ and $R^T ubw$. Assume for reductio $F \in w$. By definitions, $A \to (B \to F) \in a$, $A \in b$, $B \in b$.

As $(A \to (B \to F)) \to ((A \wedge B) \to F)$ is a theorem, $(A \wedge B) \to F \in a$. Since $A \wedge B \in b$ and $R^C abc$, we have

$F \in c$, which is impossible $c$ being consistent. Therefore, $w$ is consistent (cf. Lemma 2). Now, $u$ and $w$ are

extended to $x \in K^C$ and $y \in S^C$ such that $R^C abx$ and $R^C xby$ [Apply Lemma 1 and Lemma 6].

P10. $a \notin S^C \Rightarrow (not - R^C abc$ or $A \in c)$ [for any wff $A$]

Suppose $F \in a$ (cf. Lemma 2), $R^C abc$ and $A \in b$ (since $b$ is non-null). Let $B$ be any wff. By A13, $A \to B \in a$. Thus, $B \in c$.

LEMMA 8. *If* $\nvdash_{LC^o} A$, *then there is some* $x \in K^C$ *such that* $A \notin x$.

PROOF. By Lemma 1.

From Lemmas 3, 4 and 7 it follows that the canonical model is in fact a model. Then, completeness follows by Lemma 8. We finish this paper with the following

## 9. Note

If A12' is deleted from $LC^o$, the result is LC with the C.A.P. and minimal negation [$LC^o$m]. If, accordingly, P10 is deleted from $LC^o$ models, would presumably obtain $LC^o$m models, that is, complete semantics for LC with the C.A.P. and minimal negation.

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