A MODIFIED SUBFORMULA PROPERTY FOR THE MODAL LOGICS $K5$ AND $K5D$

Abstract

The sequent calculus $GK5$ ($GK5D$) for the modal propositional logic $K5$ ($K5D$) is presented, and it is shown that, every provable sequent $\Gamma \rightarrow \Theta$ in $GK5$ ($GK5D$) has a $GK5$-proof ($GK5D$-proof) such that every formula occurring in it is either a subformula of some formulas in $\Gamma, \Theta$, or the formula $\Box\neg\Box B$ or $\neg\Box B$, where $\Box B$ occurs in the scope of some occurrence of $\Box$ in some formulas of $\Gamma, \Theta$. Some corollaries including the interpolation property for $K5$ ($K5D$) follow from this.

1. Introduction

Think of the modal propositional logics which are obtained from the least normal logic $K$ by adding axioms among the following ones.

\begin{align*}
T & : \Box p \supset p \\
4 & : \Box p \supset \Box \Box p \\
5 & : \neg\Box p \supset \Box \neg\Box p \\
D & : \Box p \supset \neg\Box\neg p \\
B & : \neg p \supset \Box \neg\Box p
\end{align*}

Fifteen mutually distinct logics come out of $32 (= 2^5)$ ways of choice of these axioms by the following properties.

- $D$ is provable in $KT$.
- $5$ is provable in $K4B$.
- $4$ is provable in $K5B$.
- $4$ and $B$ are provable in $KT5$.
- $T$ is provable in $K4DB$.

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We divide them into the following three groups.

**Group 1:** $K$, $KT$, $KD$, $K4$, $K4D$, $S4$ ($= KT4$), $K45$, $K45D$

**Group 2:** $KB$, $KTB$, $KDB$, $K4B$, $S5$ ($= KT4B = KT5$)

**Group 3:** $K5$, $K5D$

The logics in Group 1 have sequent calculi with the cut-elimination property (and so the subformula property). Most of these facts are classical, while the proofs for $K45$ and $K45D$ are given in Shvarts [5]. Those in Group 2, on the other hand, have sequent calculi with the subformula property but without the cut-elimination property (Takano [6]).

The objects of this paper are the logics in Group 3, namely, $K5$ and $K5D$. The former is formulated as the sequent calculus $GK5$ which is obtained from the propositional fragment of $LK$ by adding the rule $$(\Box)S$$ below, and the latter as the calculus $GK5D$ which is obtained from $GK5$ by adding the rule $$(\Box)D$$ below.

$$(\Box)S: \quad \Gamma \rightarrow \Box \Theta, A \quad (\Box)D: \quad \Gamma \rightarrow \Box \Gamma$$

They are probably the most natural sequentical versions of the logics. It is to be remarked that $GK5D$ admits the following rule $$(\Box)S_D$$; in fact, when $\Theta$ is nonempty, the lower sequent of the rule is obtained from the upper one by successive application of “weakening”, $$(\Box)S$$ and “contraction”.

$$(\Box)S_D: \quad \Gamma \rightarrow \Box \Theta \quad \Box \Gamma \rightarrow \Box \Theta$$

But unfortunately, both $GK5$ and $GK5D$ lack the subformula property. In fact, it is easy to see that the sequent $\Box \Box p \rightarrow \Box \Box \Box p$ doesn’t have a $GK5D$-proof which is constituted solely of $\Box \Box \Box p, \Box \Box p, \Box p$ and $p$, where $p$ is a propositional variable. On the other hand, it has the following $GK5$-proof, where applications of the structural rules “weakening”, “contraction” and “exchange” are neglected.
\[
\begin{align*}
\square p \to \square p & \quad \to \square p, \neg \square p \\
\to \square p, \neg \square p & \quad \neg p \to \square p \\
\to \square \neg p, \square \neg p & \quad \neg \neg p, \square p \to \square \square p \\
\to \square \neg p, \square \neg p & \quad \neg \neg p, \square p \to \square \square p \\
\square \square p \to \square \square \square p
\end{align*}
\]

In view of the above example, we extend the notion of subformula to define that of $K5$-subformula.

**Definition 1.** (1) An internal subformula of a formula $A$ is a subformula of some formula $C$ such that $\square C$ is a subformula of $A$.

(2) A $K5$-subformula of a formula $A$ is either a subformula of $A$ or the formula of the form $\square \neg \square B$ or $\neg \square B$, where $\square B$ is an internal subformula of $A$.

Since $\square p$ is an internal subformula of $\square \square p$ (and also of $\square \square \square p$), both $\square \neg \square p$ and $\neg \square p$ are $K5$-subformulas of $\square \square p$ (and also of $\square \square \square p$). So, every formula which occurs in the above $GK5$-proof is a $K5$-subformula of $\square \square p$ or $\square \square \square p$. Generalizing this to any $GK5$-proof and $GK5D$-proof, we obtain the following theorem.

**Theorem.** Every provable sequent $\Gamma \to \Theta$ in $GK5$ ($GK5D$) has a $GK5$-proof ($GK5D$-proof) such that every formula occurring in it is a $K5$-subformula of some formulas in $\Gamma$ or $\Theta$.

A semantical proof of this theorem is given in the following section. It is essential for our proof that $K5$ ($K5D$) is characterized by the class of Kripke frames with Euclidean (Euclidean and serial) accessibility relations (cf. Goré [3], for example); where a binary relation $R$ on a set $W$ is called Euclidean, when $aRb$ and $aRc$ imply $bRc$ for every $a, b, c \in W$. The finite model property and so the decidability of these logics soon follow from our proof of Theorem. This slightly sharpens the proof of Chagrov-Zakharyaschev [1, Theorem 5.35], in which they filtrate a countermodel for a formula $A$ based on a Euclidean frame through the set of subformulas of $A$ as well as formulas of the form $\square \neg \square B$ or $\neg \square B$, where $\square B$ is a subformula of $A$.

The interpolation property for $K5$ and $K5D$ can be derived from our theorem along the so-called “Maehara method” (cf. Takeuti [7] and Ono [4]). Similarly, one obtains Halldén completeness of $K5D$: If $A \lor B$ is
provable in $K5D$, and if no propositional variable occurs in $A$ and $B$ in common, then $A$ or $B$ is provable in $K5D$. Note that $K5$ lacks the Halldén completeness property in contrast with $K5D$; in fact, $\Box p \lor \neg \Box \neg(q \supset q)$ is provable in $K5$, but neither $\Box p$ nor $\neg \Box \neg(q \supset q)$ is not.

By transforming proofs in the sequent calculi with the modified subformula property into ones in the Hilbert style calculi, a version of Fitting’s subformula results [2] for $K5$ and $K5D$ follows also: Whenever $A$ is a theorem of $K5$ ($K5D$), it is obtained by zero or more applications of modus ponens and necessitation from theorems of $K$ ($KD$) and formulas of the form $\neg \Box B \supset \Box \neg \Box B$, where $\Box B$ is a $K5$-subformula of $A$.

For sequent calculi, see Takeuti [7], for example. In what follows, the antecedent $\Gamma$ and the succedent $\Theta$ of the sequent $\Gamma \rightarrow \Theta$ are considered as finite sets of formulas, for simplicity.

2. Proof of Theorem

Let $G$ be either $GK5$ or $GK5D$. Throughout this section, we suppose that the sequent $\Gamma \rightarrow \Theta$ does not have a $G$-proof such that every formula occurring in it is contained in $\Xi$, where $\Xi$ denotes the set of $K5$-subformulas of some formulas in $\Gamma \cup \Theta$. Note that $\Xi$ is finite, and is closed under subformulas. We will construct a Kripke model $\langle W, R, V \rangle$ with the Euclidean accessibility relation $R$ and $o \in W$ such that, $o \models A$ for every $A \in \Gamma$ but $o \not\models B$ for every $B \in \Theta$, and moreover, $R$ is serial if $G$ is $GK5D$. This implies by the soundness of $G$ with respect to Kripke semantics that $\Gamma \rightarrow \Theta$ is unprovable in $G$, and so ends the proof of our theorem.

**Definition 2.** (1) A sequent is $\Xi$-provable (in $G$), if it has a $G$-proof which is constituted solely of formulas in $\Xi$.

(2) A set $\Delta$ of formulas is $\Xi$-saturated (in $G$), if $\Delta \subseteq \Xi$ but the sequent $\Delta \rightarrow \Delta^c$ is $\Xi$-unprovable, where $\Delta^c$ denotes the complement of $\Delta$ with respect to $\Xi$.

We denote $\Xi$-saturated sets as $a, b, c, \ldots$. It is easily seen by the logical rules for $\land, \lor, \supset$ and $\neg$, that the following proposition holds.

**Proposition 1.** The following properties hold for every $\Xi$-saturated set $a$.

(1) If $A \land B \in a$, then $A \in a$ and $B \in a$. 
A Modified Subformula Property for the Modal Logics $K5$ and $K5D$

(2) If $A \land B \in a^c$, then $A \in a^c$ or $B \in a^c$.
(3) If $A \lor B \in a$, then $A \in a$ or $B \in a$.
(4) If $A \lor B \in a^c$, then $A \in a^c$ and $B \in a^c$.
(5) If $A \lor B \in a$, then $A \in a^c$ or $B \in a$.
(6) If $A \lor B \in a^c$, then $A \in a$ and $B \in a^c$.
(7) If $\neg A \in a$, then $A \in a^c$.
(8) If $\neg A \in a^c$, then $A \in a$.

Owing to the presence of the cut rule, we can add each formula from $(\Delta \cup \Lambda)^c$ either to the antecedent or to the succedent, keeping its $\Xi$-unprovability. Thus we have the following.

Proposition 2. If $\Delta \cup \Lambda \subseteq \Xi$ but the sequent $\Delta \rightarrow \Lambda$ is $\Xi$-unprovable, there is a $\Xi$-saturated set $a$ such that $\Delta \subseteq a$ and $\Lambda \subseteq a^c$.

Since $\Gamma \cup \Theta \subseteq \Xi$ but $\Gamma \rightarrow \Theta$ is $\Xi$-unprovable by the assumption, there is a $\Xi$-saturated set $a$ such that $\Gamma \subseteq a$ and $\Theta \subseteq a^c$ by Proposition 2. Define $W_1 = \{a\}$; while let $W_2$ be the set of $\Xi$-saturated sets $b$ such that for every $B$,

$$\Box \neg \Box B \in o \text{ implies } \Box B \in b^c; \quad \text{(a)}$$
$$\Box \neg \Box B \in o^c \text{ implies } B \in b; \quad \text{(b)}$$
$$\Box \neg \Box B \in o^c \text{ implies } \Box \neg \Box B \in b^c. \quad \text{(c)}$$

Now, the Kripke frame $\langle W, R \rangle$ is defined as follows: $W$ is the direct sum $W_1 + W_2$ of $W_1$ and $W_2$; while $aRb$ iff, either $a \in W_1$, $b \in W_2$ and for every $B$,

$$\Box B \in a \text{ implies } B \in b, \quad \text{(d)}$$

or $a, b \in W_2$. Note that, when the $\Xi$-saturated set $a$ happens to satisfy both of the conditions (a) and (b), we distinguish two $a$'s, namely, the sole element of $W_1$ and an element of $W_2$. If $aRb$ and $aRc$, then $b, c \in W_2$ and so $bRc$; hence $R$ is Euclidean. Note here that $W$ is finite.

Proposition 3. If $a \in W$ and $\Box A \in a^c$, then $A \in b^c$ and $aRb$ for some $b \in W_2$.

Proof. Put $\Delta = \{B|\Box B \in a\}$ and $\Lambda = \{B|\Box B \in a^c\}$. Since $\Delta \cup \Box \Lambda \cup \{A\} \subseteq \Xi$ but the sequent $\Delta \rightarrow \Box \Lambda, A$ is $\Xi$-unprovable by $(\Box)_5$ and the structural rules, $\Delta \subseteq b$ and $\Box \Lambda \cup \{A\} \subseteq b^c$ for some $\Xi$-saturated set $b$ by Proposition 2.
We first prove \( b \in W_2 \) by showing (a), (b) and (c). (a) Suppose \( \Box \neg \Box B \in o \). If \( a \in W_1 \), then \( \neg \Box B \in \Delta \subseteq b \), so \( \Box B \in b' \) by Proposition 1 (7). If \( a \in W_2 \) on the other hand, \( \Box B \in a' \) by (a) for \( a \), so \( \Box B \in \Delta \subseteq b' \).

(b) Suppose \( \Box \neg \Box B \in o' \). Then \( \Box \neg \Box B \in a' \) irrespective of whether \( a \in W_1 \) or \( a \in W_2 \), by (c) for \( a \) when \( a \in W_2 \). If \( \Box B \not\in a \), since the sequent \( \rightarrow \Box B, \neg \Box B \) is \( \Sigma \)-provable, \( a \rightarrow a' \) is \( \Sigma \)-provable by the structural rules, which is a contradiction. Hence \( \Box B \in a \), so \( B \in \Delta \subseteq b \).

(c) Suppose \( \Box \neg \Box B \in o' \). Similarly to the proof of (b), \( \Box \neg \Box B \in a' \), so \( \Box \neg \Box B \in \Delta \subseteq b' \).

Next we prove \( aRb \). Suppose first that \( a \in W_1 \). If \( \Box B \in a \), then \( B \in \Delta \subseteq b \). Hence (d) holds for every \( B \), and so \( aRb \). If \( a \in W_2 \) on the other hand, \( aRb \) clearly.

The following proposition is proved similarly by utilizing \( (\Box)^5_D \) instead of \( (\Box)^5 \); so \( R \) is serial, if \( G \) is \( GK5D \).

**Proposition 4.** Suppose that \( G \) is \( GK5D \). If \( a \in W \), then \( aRb \) for some \( b \in W_2 \). ■

Next, the valuation function \( V \) based on the Kripke frame \( \langle W, R \rangle \) is defined by: \( V(p) = \{ a \in W | p \in a \} \) for every propositional variable \( p \). Thus we have defined the Kripke model \( \langle W, R, V \rangle \) with the Euclidean accessibility relation \( R \), and moreover, \( R \) is serial if \( G \) is \( GK5D \). Let \( |= \) be the satisfaction relation derived from \( \langle W, R, V \rangle \).

**Proposition 5.** Suppose \( a \in W_2 \) and \( C \) is an internal subformula of some formulas in \( \Gamma \cup \Theta \).

1. If \( C \in a \), then \( a |= C \).
2. If \( C \in a' \), then \( a \not|= C \).

**Proof.** By simultaneous induction on the construction of \( C \).

Case 1: \( C \) is a propositional variable. Since \( a |= C \) iff \( a \in V(C) \) iff \( C \in a \), both assertions hold certainly.

Case 2: \( C \) is \( A \land B \). Recall that \( a |= C \) iff \( a |= A \) and \( a |= B \). (1) Suppose \( C \in a \). Then \( A \in a \) and \( B \in a \) by Proposition 1 (1). So \( a |= A \) and \( a |= B \) by the hypothesis of induction, and hence \( a |= C \). (2) Suppose next that \( C \in a' \). Then \( A \in a' \) or \( B \in a' \) by Proposition 1 (2). So \( a \not|= A \) or \( a \not|= B \) by the hypothesis of induction, and hence \( a \not|= C \).

Cases 3–5: \( C = A \lor B, A \supset B \) or \( \neg A \). Similar to Case 2.
Case 6: $C$ is $\Box A$. Recall that $a \models C$ iff $b \models A$ for every $b \in W$ such that $aRb$. It is crucial that $\Box \neg C \in \Sigma$. (1) Suppose $C \in a$. If $\Box \neg C \in o$, then $C \in a^c$ by (a), which is a contradiction. Hence $\Box \neg C \in o^c$. So for every $b \in W$ such that $aRb$, it follows $b \in W_2$ and $A \in b$ by (b), and therefore $b \models A$ by the hypothesis of induction. Hence $a \models C$. (2) Suppose next that $C \in a^c$. By Proposition 3, $A \in b^c$ and $aRb$ for some $b \in W_2$. By the hypothesis of induction, $b \not\models A$, so $a \not\models C$. 

**Proposition 6.** Suppose $a \in W_1$ and $C$ is a subformula of some formulas in $\Gamma \cup \Theta$.

1. If $a \models C$, then $a \models C$.
2. If $a \models a^c$, then $a \not\models C$.

**Proof.** Modify Case 6 of the proof of the previous proposition as follows.

1. Suppose $C \in a$. For every $b \in W$ such that $aRb$, it follows $b \in W_2$ and $A \in b$ by (d), so $b \models A$ by Proposition 5 (1). Hence $a \models C$. (2) Suppose next that $C \in a^c$. By Proposition 3, $\neg C \in b^c$ and $aRb$ for some $b \in W_2$. By Proposition 5 (2), $b \not\models A$, so $a \not\models C$.

By Proposition 6, for the element $o$ of $W_1$, if $A \in \Gamma$, then $A \models o$; while, if $B \in \Theta$, then $B \models o^c$, so $o \not\models B$. This completes the proof of our theorem.

Since the set $W$ of the Kripke model $\langle W, R, V \rangle$ is finite, the finite model property and so the decidability of $K5$ and $K5D$ have been obtained as byproducts.

**References**


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