Yasusi Hasimoto

FINITE MODEL PROPERTY FOR SOME INTUITIONISTIC MODAL LOGICS

Abstract

In this paper, we will show some logic has the finite model property by using filtration method. Although the filtration method for classical modal logics has been studied comprehensively, the method is not completely applied to intuitionistic modal logics yet.

1. Introduction

Classical modal logics are the classical logic with some axioms and rules for modal operators. Usually, ◻ is considered to be equal to ¬□¬ in any classical modal logic, where ¬ denotes the negation. In this paper, we will study intuitionistic modal logics, i.e., modal logics based on the intuitionistic logic \( \text{Int} \) (and its extensions). In intuitionistic modal logics the necessity operator ◻ and the possibility operator ◇ are not necessarily considered to be dual. In other words, we do not always assume that ◻p ⇔ ¬◇¬p and ◇p ⇔ ¬◻¬p hold. This provides more possibilities for introducing various kinds of intuitionistic modal logics. We will consider here intuitionistic modal logics with independent ◻ and ◇.

Let \( \mathcal{L}_{◻, ◇} \) be the language of propositional modal logic with countably many propositional variables, \( p, q, r, \ldots \) and the connectives \( \land, \lor, \rightarrow, \bot, ◻, ◇ \). The formula ◻¬α is defined as α → ⊥ and ◇ as ⊥ → ⊥. We call a subset \( L \) a logic if it contains the intuitionistic logic \( \text{Int} \) and axioms
$(\Box p \land \Box q) \rightarrow \Box(p \land q)$, $(\Diamond p \lor q) \rightarrow (\Diamond p \lor \Diamond q)$, $\Box \top$ and $\neg\Diamond \bot$, and is closed under modus ponens, substitution, and the rules $\vdash \alpha \rightarrow \beta, \vdash \Box \alpha \rightarrow \Box \beta$ and $\vdash \alpha \rightarrow \beta, \vdash \Diamond \alpha \rightarrow \Diamond \beta$. We denote by $\text{IntK}_23$ the smallest logic in the above sense. This logic was introduced in Wolter and Zakharyaschev [14]. The smallest logic containing a logic $L$ and a formula $\varphi$ is denoted by $L \oplus \varphi$, in the following.

Many ways of defining intuitionistic analogues of classical normal modal logics have been considered. First, one can take the family of logics extending $\text{IntK}_\Box$. A model theory for a logic extending $\text{IntK}_\Box$ was developed by H. Ono [9], M. Borič and K. Došen [3], V. H. Sotirov [12] and F. Wolter and M. Zakharyaschev [14]. A possibility operator $\Diamond$ in those logics can be defined in the classical way by taking $\Diamond \varphi$ as $\neg\Box \neg \varphi$. Note, however, that in general this $\Diamond$ doesn’t distribute over disjunction and that the connection via negation between $\Box$ and $\Diamond$ is too strong from intuitionistic point of view.

Another family of normal logics are logics extending $\text{IntK}_\Diamond$. These logics were studied by M. Borič and K. Došen [3], V. H. Sotirov [12] and F. Wolter [13].

Fischer Servi [6] constructed $\text{FS}$, a logic extending $\text{IntK}_{\Box \Diamond}$ with a weak connection between the necessity operator $\Box$ and possibility operator $\Diamond$. The standard translation of modal formulas into first order formulas not only embeds $K$ into classical predicate logic but also $\text{FS}$ into intuitionistic predicate logic. Various extensions of $\text{FS}$ were studied by R.A. Bull [4], H. Ono [9], G. Fischer Servi [5], [6], [7], F. Wolter and M. Zakharyaschev [14], F. Wolter [13] and C. Grefe [8]. A well-known extension of $\text{FS}$ is the logic $\text{MIPC}$ introduced by A. Prior [11]. R.A. Bull noticed that $\text{MIPC}$ is embedded into the monadic fragment of intuitionistic predicate logic. H. Ono [9], H. Ono and N.-Y. Suzuki [10] and G. Bezhanishvili [2] investigated relations between logics extending $\text{MIPC}$ and superintuitionistic predicate logics, and their models. V. H. Sotirov in [12] investigated some logics weaker logics than $\text{IntK}_3\Box$, which are not necessarily normal. He also dealt with some extensions of $\text{IntK}_3\Box$. 
2. Kripke type semantics

In this section we will consider Kripke-type semantics for intuitionistic modal logics.

A structure $\mathcal{F} = (W, \prec, R_\Box, R_\Diamond)$ is called a Kripke frame if the following conditions are satisfied. $W$ is a nonempty set, $\prec$ is a partial order on $W$, both $R_\Box$ and $R_\Diamond$ are binary relations on $W$, such that $\prec \circ R_\Diamond \circ \prec = R_\Box$, $\prec^{-1} \circ R_\Diamond \circ \prec^{-1} = R_\Diamond$.

A valuation $v$ on $\mathcal{F}$ is a homomorphism from the set of all formulas of $\mathcal{L}_\Box\Diamond$ to $\text{Up}W$ where $\text{Up}W = \{ V \subseteq W \mid (x \in V \text{ and } x \prec y) \Rightarrow y \in V \}$, with operators $\rightarrow$, $\Box$ and $\Diamond$ defined respectively as follows.

- $X \rightarrow Y := \{ w \in W \mid \text{for any } v, \ w \prec v \text{ and } v \in X \text{ implies } v \in Y \}$,
- $\Box X := \{ w \in W \mid \text{for any } v, \ w R_\Box v \text{ implies } v \in X \}$,
- $\Diamond X := \{ w \in W \mid w R_\Diamond v \text{ for some } v \in X \}$.

A pair $\mathcal{M} = (\mathcal{F}, v)$ of a Kripke frame $\mathcal{F}$ and a valuation $v$ on $\mathcal{F}$ is called a model. In this case, $\mathcal{F}$ is called the base of a model $\mathcal{M}$. For any $\alpha \in \text{Form}(\mathcal{L}_\Box\Diamond)$, any model $\mathcal{M}$ and any $x \in W$, $\alpha$ is true at $x$ in $\mathcal{M}$ (in symbols, $(\mathcal{M}, x) \models \alpha$ or simply $x \models \alpha$ if $\mathcal{M}$ is understood) if $x \in v(\alpha)$. Also, $\alpha$ is true in $\mathcal{M}$ (in symbols, $\mathcal{M} \models \alpha$) if $W = v(\alpha)$. If it is not true in $\mathcal{M}$ then it is refuted in $\mathcal{M}$. For any $\alpha \in \text{Form}(\mathcal{L}_\Box\Diamond)$ and any Kripke frame $\mathcal{F}$, $\alpha$ is valid in $\mathcal{F}$ (in symbol, $\mathcal{F} \models \alpha$) if $W = v(\alpha)$ for any valuation $v$ on $\mathcal{F}$.

Similarly to classical modal logic, we can develop the correspondence theory. Here are some examples.

**Proposition 2.1.** For any frame $\mathcal{F}$, $\mathcal{F}$ validates each formula in the following list if $\mathcal{F}$ satisfies the corresponding condition in the list.

\[
\begin{align*}
\Box(p \rightarrow q) \rightarrow (\Diamond p \rightarrow \Diamond q) & \quad y R_\Box x \Rightarrow \exists z(x \prec z \& y R_\Box z) \\
\Diamond(p \rightarrow q) \rightarrow (\Box p \rightarrow \Diamond q) & \quad y R_\Diamond x \Rightarrow \exists z(x \prec z \& y R_\Diamond z) \\
(\Diamond p \rightarrow \Box q) \rightarrow (\Box p \rightarrow q) & \quad x R_\Box y \Rightarrow \exists z(x \prec z \& z R_\Diamond y) \\
\neg \Box \bot & \quad R_\Box : \text{serial, i.e.}, \forall x \exists y x R_\Box y \\
\Diamond \top & \quad R_\Diamond : \text{serial, i.e.}, \forall x \exists y x R_\Diamond y \\
\Box p \rightarrow p & \quad R_\Box : \text{reflexive} \\
p \rightarrow \Diamond p & \quad R_\Diamond : \text{reflexive}
\end{align*}
\]
\( \Box p \to \Box \Box p \quad R_\Box : \text{transitive} \)

\( \Diamond \Diamond p \to \Diamond p \quad R_\Diamond : \text{transitive} \)

\( \Diamond \Box p \to \Box p \quad (xR_\Box y \land xR_\Box z) \Rightarrow zR_\Box y \)

\( \Diamond p \to \Diamond \Diamond p \quad (xR_\Box y \land xR_\Box z) \Rightarrow yR_\Box z \)

\( \Box (\Box p \to q) \lor \Box (\Box q \to p) \quad (xR_\Box y \land xR_\Box z) \Rightarrow (yR_\Box z \lor zR_\Box y) \)

The following is a list of some intuitionistic modal logics, which are discussed often in the literature.

\[
\begin{align*}
FS &= \text{Int}K_{\Box} \oplus \{ \Diamond (p \to q) \to (\Box p \to q), (\Box p \to \Box q) \to \Box (p \to q) \}, \\
\text{IntD}_{\Box} &= \text{Int}K_{\Box} \oplus \{ \neg \Box \bot, \Diamond \top \}, \\
\text{IntT}_{\Box} &= \text{Int}K_{\Box} \oplus \{ \Box p \to \Box p, \Diamond \top \to \Diamond \Diamond p \}, \\
\text{IntK4}_{\Box} &= \text{Int}K_{\Box} \oplus \{ \Diamond \Box p \to \Box p, \Diamond \Diamond p \to \Diamond \Diamond p \}, \\
\text{IntS4}_{\Box} &= \text{Int}K_{\Box} \oplus \{ \Box p \to \Box p, \Diamond \top \to \Diamond \Diamond p \}, \\
\text{IntS4.3}_{\Box} &= \text{Int}S5_{\Box} \oplus \{ \Box p \to \Box p, \Diamond \top \to \Diamond \Diamond p \}, \\
\text{IntS5}_{\Box} &= \text{Int}K_{\Box} \oplus \{ \Box p \to \Box p, \Diamond \top \to \Diamond \Diamond p \}, \\
\text{MIPC} &= \text{Int}S5_{\Box} \oplus \{ \Box p \to \Box p, \Diamond \top \to \Diamond \Diamond p \}, \\
\end{align*}
\]

The conditions for frames validating these logics are given below.

\[
\begin{align*}
FS & \quad yR_\Box x \Rightarrow \exists z(x < z \land yR_\Box z \land yR_\Box y) \\
& \quad xR_\Box y \Rightarrow \exists z(x < z \land zR_\Box y \land zR_\Box y) \\
\text{IntD}_{\Box} & \quad R_\Box, R_\Diamond: \text{serial} \\
\text{IntT}_{\Box} & \quad R_\Box, R_\Diamond: \text{reflexive} \\
\text{IntK4}_{\Box} & \quad R_\Box, R_\Diamond: \text{transitive} \\
\text{IntS4}_{\Box} & \quad R_\Box, R_\Diamond: \text{reflexive and transitive} \\
\text{IntS4.3}_{\Box} & \quad R_\Box, R_\Diamond: \text{reflexive and transitive} \\
& \quad (xR_\Box y \land xR_\Box z) \Rightarrow (yR_\Box z \lor zR_\Box y) \\
\text{IntK5}_{\Box} & \quad (xR_\Box y \land xR_\Box z) \Rightarrow (zR_\Box y \land yR_\Box z) \\
\text{IntS5}_{\Box} & \quad R_\Box = R_\Box^{-1} \text{ and } R_\Box: \text{reflexive and transitive} \\
\text{MIPC} & \quad R_\Box = R_\Box^{-1} \text{ and } R_\Box: \text{reflexive and transitive} \\
& \quad xR_\Box y \Rightarrow \exists z(x < z \land yR_\Box z \land zR_\Box y) \\
\end{align*}
\]

A logic \( L \) is called **Kripke complete** if there is a class \( \mathcal{C} \) of Kripke frames such that \( L \vdash \alpha \iff \mathcal{F} \models \alpha \) for any \( \mathcal{F} \in \mathcal{C} \). In order to show that a given logic \( L \) is Kripke complete, it is sufficient that the canonical frame validates
We can show that the following logics are Kripke complete, by using canonical frames (see [12], [14]).

**Proposition 2.2.** Logics $\text{IntK}_{2\cup}$, $\text{FS}$, $\text{IntD}_{2\cup}$, $\text{IntT}_{2\cup}$, $\text{IntK4}_{2\cup}$, $\text{IntS4}_{2\cup}$, $\text{IntS4.3}_{2\cup}$, $\text{IntK5}_{2\cup}$, $\text{IntS5}_{2\cup}$ and $\text{MIPC}$ are Kripke complete.

### 3. Filtration method

A logic $L$ has the **finite model property** if for every non-theorem $\varphi$ of $L$, there exists a finite frame $F$ such that $F \models L$ and $F \not\models \varphi$. As is well known, if it is moreover **finitely axiomatizable**, i.e., $L = \text{IntK}_{2\cup} \oplus \Gamma$ for some finite set $\Gamma$ of formulas, then it is decidable.

By using filtration method, we will show in the following that many extensions of $\text{IntK}_{2\cup}$ have the finite model property. Let $M$ be a model and $\Sigma$ be a set of formulas closed under subformulas, i.e., $\text{Sub}\varphi \subseteq \Sigma$ whenever $\varphi \in \Sigma$, where $\text{Sub}\varphi$ is the set of all subformulas of $\varphi$. Define an equivalence relation $\sim_{\Sigma}$ on $W$, by taking

$$x \sim_{\Sigma} y \iff (M, x) \models \varphi \text{ iff for every } \varphi \in \Sigma, (M, y) \models \varphi,$$

and say that $x, y$ are $\Sigma$-equivalent in $M$. Denote by $[x]_{\Sigma}$ the equivalence class generated by $x$. If $\Sigma$ is understood from context, $[x]_{\Sigma}$ is written simply $[x]$.

A model $M_{\Sigma} = (W_{\Sigma}, \ll_{\Sigma}, R_{\circlearrowleft \downarrow \Sigma}, R_{\circlearrowright \downarrow \Sigma}, v_{\Sigma})$ is called a **filtration** of $M$ through $\Sigma$ if the following conditions are satisfied.

1. $W_{\Sigma} = \{ [x] \mid x \in W \}$
2. $v_{\Sigma}(p) = \{ [x] \mid x \in v(p) \}$, for every propositional variable $p \in \Sigma$,
3. for all $x, y \in W$ $x \ll y$ implies $[x] \ll_{\Sigma} [y]$,
4. for all $x, y \in W$ $xR_{\circlearrowleft y}$ implies $[x]R_{\circlearrowleft \downarrow \Sigma}[y]$,
5. for all $x, y \in W$ $xR_{\circlearrowright y}$ implies $[x]R_{\circlearrowright \downarrow \Sigma}[y]$,
6. for $x, y \in W$ and $\varphi \in \Sigma$ if $[x] \ll_{\Sigma} [y]$ then $y \models \varphi$ whenever $x \models \varphi$,
7. for $x, y \in W$ and $\Box \varphi \in \Sigma$ if $[x]R_{\circlearrowleft \downarrow \Sigma}[y]$ then $y \models \varphi$ whenever $x \models \Box \varphi$,
8. for $x, y \in W$ and $\Diamond \varphi \in \Sigma$ if $[x]R_{\circlearrowright \downarrow \Sigma}[y]$ then $x \models \Diamond \varphi$ whenever $y \models \varphi$. 

**Finite Model Property for Some Intuitionistic Modal Logics**
Then, we can show the following.

**Theorem 3.1.** Let \( \mathcal{M}_\Sigma \) be a filtration of a model \( \mathcal{M} \) through a set \( \Sigma \) of formulas. Then for every \( x \) in \( \mathcal{M} \) and every formula \( \varphi \in \Sigma \),

\[
(\mathcal{M}, x) \models \varphi \iff (\mathcal{M}_\Sigma, [x]) \models \varphi.
\]

In general, the conditions from (3) to (8) do not determine the binary relations \( \mathcal{A}_\Sigma, \mathcal{R}_\Sigma, \mathcal{R}_\Sigma^{-1} \) uniquely. Actually, they allow us to choose any relations \( \mathcal{A}_\Sigma, \mathcal{R}_\Sigma, \mathcal{R}_\Sigma^{-1} \) such that \( \mathcal{A}_\Sigma \subseteq \mathcal{A}_\Sigma \subseteq \mathcal{B}_\Sigma, \mathcal{R}_\Sigma \subseteq \mathcal{R}_\Sigma \subseteq \mathcal{R}_\Sigma^{-1}, \mathcal{R}_\Sigma^{-1} \), where

\[
\begin{align*}
\mathcal{A}_\Sigma &= \{ ([x], [y]) \mid \exists x', y'(x \sim \Sigma x' & y \sim \Sigma y' & x' \sim y') \}, \\
\mathcal{B}_\Sigma &= \{ ([x], [y]) \mid \exists x', y'(x \sim \Sigma x' & y \sim \Sigma y' & x' \sim y' R_\Sigma y') \}, \\
\mathcal{R}_{\Sigma} &= \{ ([x], [y]) \mid \exists x', y'(x \sim \Sigma x' & y \sim \Sigma y' & x' \sim R_\Sigma y') \}, \\
\mathcal{R}_{\Sigma}^{-1} &= \{ ([x], [y]) \mid \forall \varphi \in \Sigma (x \models \varphi \Rightarrow y \models \varphi) \}, \\
\mathcal{R}_{\Sigma}^{-1} &= \{ ([x], [y]) \mid \forall \varphi \in \Sigma (y \models \varphi \Rightarrow x \models \varphi) \}.
\end{align*}
\]

Indeed, if all of \( [x] \mathcal{A}_\Sigma [y], [x] \mathcal{R}_{\Sigma}^{-1} [y] \) and \( [x] \mathcal{R}_{\Sigma} [y] \) hold, then by (6), (7) and (8), \( [x] \mathcal{B}_\Sigma [y], [x] \mathcal{R}_{\Sigma} [y] \) and \( [x] \mathcal{R}_{\Sigma}^{-1} [y] \) hold, respectively. Also if \( [x] \mathcal{A}_\Sigma [y], [x] \mathcal{R}_{\Sigma}^{-1} [y] \) and \( [x] \mathcal{R}_{\Sigma} [y] \) then \( x' \sim y', x' R_{\Sigma} y' \) and \( x' R_{\Sigma} y' \) for some \( x' \in [x], y' \in [y] \). Hence, by (3), (4) and (5), \( [x] \mathcal{A}_\Sigma [y], [x] \mathcal{R}_{\Sigma}^{-1} [y] \) and \( [x] \mathcal{R}_{\Sigma} [y] \), respectively. We can show that \( \mathcal{A}_\Sigma \) is a partial order, and that both of \( \mathcal{B}_\Sigma \circ \mathcal{R}_{\Sigma} \circ \mathcal{A}_\Sigma = \mathcal{R}_{\Sigma}^{-1}, \mathcal{A}_\Sigma \circ \mathcal{R}_{\Sigma} \circ \mathcal{A}_\Sigma = \mathcal{R}_{\Sigma}^{-1} \) hold. Therefore, \( (W_\Sigma, \mathcal{A}_\Sigma, \mathcal{B}_\Sigma, \mathcal{R}_\Sigma, \mathcal{R}_\Sigma^{-1}) \) is a frame.

On the other hand, not all of triples \( \langle \mathcal{A}_\Sigma, \mathcal{R}_{\Sigma}, \mathcal{R}_{\Sigma}^{-1} \rangle \) in these intervals give rise to filtrations of frames. More precisely, the reflexivity and the anti-symmetry of \( \mathcal{A}_\Sigma \) hold always, while \( \mathcal{A}_\Sigma \) may not be transitive, and \( \mathcal{A}_\Sigma \circ \mathcal{R}_\Sigma \circ \mathcal{A}_\Sigma = \mathcal{R}_{\Sigma}^{-1} \) and \( \mathcal{A}_\Sigma \circ \mathcal{R}_\Sigma \circ \mathcal{A}_\Sigma = \mathcal{R}_{\Sigma}^{-1} \) may fail to hold. To get a transitive relation it is enough to take the transitive closure \( \mathcal{A}_\Sigma \) for \( \mathcal{A}_\Sigma \). Clearly, \( \mathcal{A}_\Sigma \) satisfies (3). By the transitivity of \( \mathcal{B}_\Sigma \), \( \mathcal{A}_\Sigma \) satisfies (6). We define also \( \mathcal{R}_{\Sigma}^+ \) and \( \mathcal{R}_{\Sigma}^- \) by
\( R_2^\circ \Sigma = \Delta_\Sigma \circ R_2 \circ \Delta_\Sigma \)

and

\( R_3^\circ \Sigma = \Delta_\Sigma^{-1} \circ R_3 \circ \Delta_\Sigma^{-1} \).

Then, it is easily shown that they satisfy (4), (5), (7) and (8). Therefore, \((W_\Sigma, \Delta_\Sigma, R_2, R_3)\) is also a frame. Clearly, if \((W_\Sigma, \Delta_\Sigma, R_2, R_3)\) is a frame, \( \Delta_\Sigma \circ R_2 = R_2 \circ \Delta_\Sigma \) and \( \Delta_\Sigma \circ R_3 = R_3 \circ \Delta_\Sigma \) hold. Therefore, for any filtration \((W_\Sigma, \Delta_\Sigma, R_2, R_3)\), we have \( \Delta_\Sigma \subseteq \Delta_\Sigma \subseteq \Delta_\Sigma \), \( R_2 \subseteq R_2 \subseteq R_2 \) and \( R_3 \subseteq R_3 \subseteq R_3 \).

The filtration on the frame \( \overline{F}_\Sigma = (W_\Sigma, \Delta_\Sigma, R_2, R_3) \) is called the finest filtration of \( M \) through \( \Sigma \). Also, the filtration on the frame \( \overline{F}_\Sigma = (W_\Sigma, \Delta_\Sigma, R_2, R_3) \) discussed above is called the coarsest filtration of \( M \) through \( \Sigma \).

If \( \Sigma \) is finite then \( W_\Sigma \) is finite (in fact, it contains at most \( 2^{\mid \Sigma \mid} \) elements). Therefore, to prove the finite model property of a logic \( L \), it suffices to show that for every non-theorem \( \varphi \) of \( L \) and a model \( M \) of \( L \) such that \( M \not\models \varphi \), if there exists a filtration of \( M \) through a finite set \( \Sigma \) containing \( \varphi \) such that \( F_\Sigma \models L \). If this is really the case then we say that \( L \) admits filtration.

Suppose that \( \mathcal{P} \) is a property of frames. Suppose moreover that a given logic \( L \) is sound with respect to the class \( \mathcal{C} \) of frames satisfying a property \( \mathcal{P} \), and that the canonical frame of \( L \) satisfies \( \mathcal{P} \). In such a case, to prove that \( L \) has finite model property it suffices to show that for each non-theorem \( \varphi \) of \( L \), there exists a finite set \( \Sigma \) containing \( \varphi \) such that a filtration \( F_\Sigma \) of any \( M \) in \( \mathcal{C} \) through \( \Sigma \) satisfies \( \mathcal{P} \).

**Theorem 3.2.** Any of \( \text{IntK}_{\ominus \ominus} \), \( \text{IntD}_{\ominus \ominus} \), \( \text{IntT}_{\ominus \ominus} \), \( \text{IntK4}_{\ominus \ominus} \), \( \text{IntS4}_{\ominus \ominus} \) and \( \text{IntS5}_{\ominus \ominus} \) admits filtration and hence has the finite model property.

**Proof.** Clearly, \( \text{IntK}_{\ominus \ominus} \) admits filtration. For both \( \text{IntD}_{\ominus \ominus} \) and \( \text{IntT}_{\ominus \ominus} \), any filtration works well.

For \( \text{IntK4}_{\ominus \ominus} \), suppose that \( M \) is a model with transitive relations \( R_2 \) and \( R_3 \).

First, we will consider the finest filtration. Let \( \Sigma \) be a set of formulas closed under subformulas. We take the transitive closures \( (R_2^\circ \Sigma)^\infty \) and \( (R_3^\circ \Sigma)^\infty \) of \( R_2^\circ \Sigma \) and \( R_3^\circ \Sigma \). It is easily shown that they satisfy (4) and
(5). We will show that \((R^\ast_0 \Sigma)\) satisfies (7). Suppose that \([x]R_0 \Sigma[y]\) and \(\Box \psi \in \Sigma\). Then, there exist \(x', y'\) such that \(x' \in [x], y' \in [y]\) and \(x'R_0 y'\). If \(x \models \Box \psi\), then \(x' \models \Box \psi\). Since \(R_0\) is transitive, \(y' \models \Box + \psi\). Hence \(y \models \Box + \psi\). By iterating this argument, we have that for any \(\Box \psi \in \Sigma\), if \([x](R^\ast_0 \Sigma)\) and \(x \models \Box \psi\), then \(y \models \Box + \psi\). Thus, \((R^\ast_0 \Sigma)\) satisfies (7). Similarly, we can show that \((R^\ast_0 \Sigma)\) satisfies (8). Thus, the frame \((W_\Sigma, <, \Sigma_0, (R^\ast_0 \Sigma), (R_0 \Sigma)\) is both a filtration and a frame for \(\text{IntK4}_0\).

Next, we will consider the coarsest filtration. Let us take a set
\[
\Sigma_0 = \text{Sub}_\varphi \cup \{\square \Box \psi \mid \Box \psi \in \text{Sub}_\varphi\} \cup \{\diamond \diamond \psi \mid \psi \in \text{Sub}_\varphi\}.
\]

We will check that in the coarsest filtration both \(R_0 \Sigma\) and \(R_0 \Sigma_0\) are transitive.

Suppose first that \([x]R_0 \Sigma_0[y], [y]R_0 \Sigma_0[z]\) and \(\Box \psi \in \Sigma_0\). When \(\Box \psi \in \text{Sub}_\varphi\), if \(x \models \Box \psi\), then \(x \models \Box \Box \psi\) by the transitivity. Since \(\Box \Box \psi \in \Sigma_0\), \(y \models \Box \psi\). Hence, \(z \models \Box \psi\). Thus, \([x]R_0 \Sigma_0[z]\). Otherwise, \(\Box \psi \subseteq \Box \Box \psi \in \{\Box \Box \psi \mid \Box \psi \in \text{Sub}_\varphi\}\) for some \(\chi\). If \(x \models \Box \Box \psi\), then \(y \models \Box \chi\), since \(\Box \Box \psi \in \Sigma_0\). By the transitivity, \(y \models \Box \chi\). Hence, \(z \models \Box \chi\). Thus, \([x]R_0 \Sigma_0[z]\).

Next suppose that \([x]R_0 \Sigma_0[y], [y]R_0 \Sigma_0[z]\) and \(\Box \psi \in \Sigma_0\). When \(\Box \psi \in \text{Sub}_\varphi\), if \(z \models \psi\), then \(y \models \Box \psi\). Since \(\Box \psi \in \Sigma_0\), \(x \models \Box \psi\). Hence, \(x \models \Box \psi\) by the transitivity. Thus, \([x]R_0 \Sigma_0[z]\). Otherwise, \(\Box \psi = \Box \chi\in \{\Box \chi \mid \Box \psi \in \text{Sub}_\varphi\}\) for some \(\chi\). If \(z \models \Box \psi\), then \(y \models \Box \psi\). By the transitivity, \(y \models \Box \chi\). Since \(\Box \chi \in \Sigma_0\), \(x \models \Box \chi\). Therefore, \([x]R_0 \Sigma_0[z]\).

Thus, the frame \((W_\Sigma, <, \Sigma_0, (R_0 \Sigma_0), (R_0 \Sigma_0)\) is also a frame for \(\text{IntK4}_0\).

Therefore, for any filtration \((W_\Sigma, <, \Sigma_0, (R_0 \Sigma_0), (R_0 \Sigma_0)\), we have \(\Sigma_0 \subseteq \Sigma_0 \subseteq \Sigma_0 \subseteq (R_0 \Sigma_0) \subseteq (R_0 \Sigma_0)\) and \(R_0 \Sigma_0 \subseteq (R_0 \Sigma_0) \subseteq (R_0 \Sigma_0)\). Thus, we have shown that the frame \((W_\Sigma, <, \Sigma_0, (R_0 \Sigma_0), (R_0 \Sigma_0))\) is also a filtration and a frame for \(\text{IntK4}_0\), for any given filtration \((W_\Sigma, <, \Sigma_0, (R_0 \Sigma_0), (R_0 \Sigma_0))\).

We take \(\text{IntS4}_0\) next. When \(R_0\) and \(R_0\) are reflexive, by (4) and (5) \(R_0 \Sigma\) and \(R_0 \Sigma_0\) in any filtration are reflexive, respectively. The filtrations for \(\text{IntK4}_0\) work well also for \(\text{IntS4}_0\). Moreover, \((R^\ast_0 \Sigma) = R^\infty_0 \Sigma\) and \((R^\ast_0 \Sigma) = R^\infty_0 \Sigma\) by the reflexivities of \(R_0 \Sigma\) and \(R_0 \Sigma_0\).
We can show that $\textbf{IntS5}_\odot$ admits filtration, similarly to the case of $\textbf{IntK4}_\odot$. We will take here the following set, which we call $\Sigma_1$,

$$\text{Sub}_\varphi \cup \{\square \square \psi, \square \psi \mid \psi \in \text{Sub}_\varphi\} \cup \{\Diamond \Diamond \psi, \square \Diamond \psi \mid \psi \in \text{Sub}_\varphi\},$$

instead of $\Sigma_0$ when considering the coarsest filtration: Then, we can show that for any filtration $(W_{\Sigma_1}, \triangleleft_{\Sigma_1}, R_2^{\Sigma_1}, R_3^{\Sigma_1}, v_{\Sigma_1})$, we have $\triangleleft_{\Sigma_1} \subseteq \triangleleft^\infty_{\Sigma_1}$ and $R_2^{\Sigma_1} \subseteq (R_2^{\Sigma_1} \circ R_3^{\Sigma_1})^\infty \subseteq (R_2^{\Sigma_1} \circ R_3^{\Sigma_1})^{-1} \subseteq \overline{R_2^{\Sigma_1}} \subseteq (R_2^{\Sigma_1} \circ R_3^{\Sigma_1})^\infty \subseteq (R_2^{\Sigma_1} \circ R_3^{\Sigma_1})^{-1} \subseteq \overline{R_2^{\Sigma_1}}$. Therefore, the frame $(W_{\Sigma_1}, \triangleleft_{\Sigma_1}, (R_2^{\Sigma_1} \circ R_3^{\Sigma_1})^\infty, (R_2^{\Sigma_1} \circ R_3^{\Sigma_1})^{-1})$ is both a filtration and a frame $\textbf{IntS5}_\odot$, for any given filtration $(W_{\Sigma_1}, \triangleleft_{\Sigma_1}, R_2^{\Sigma_1}, R_3^{\Sigma_1})$. 

4. Final remarks

In [12], V. H. Sotirov proved already the finite model property for $\textbf{IntK}_\odot$, $\textbf{IntT}_\odot$, $\textbf{IntS4}_\odot$ and $\textbf{IntS5}_\odot$, by using the coarsest filtrations. In the previous section, we have shown that more general filtration method can be applied to get a proof of the finite model property of these logics. We note here that C. Greffe [8] showed the finite model property of $\textbf{FS}$, and that T. Aoto and H. Shirasu [1] discussed the finite model property of intuitionistic modal logics over $\textbf{MIPC}$.

In general, the finite model property for bimodal logics is much more difficult than that for mono-modal logics. Many problems remain on the finite model property for intuitionistic modal logics yet.

To finish off let us remark that if we restrict the modal operators only to $\square$ operator, some otherwise problematic logics will admit filtration. For example, though we fail to prove that $\textbf{IntS4.3}_\odot$ admits filtration, $\textbf{IntS4.3}_\odot$, where the only modal operator is $\square$, admits filtration, because we can take a $\square$-rooted counter-model.

The present author would like to thank my adviser Hiroakira Ono for his helpful comments and suggestions. He would like to thank also Tomasz Kowalski for reading the previous version and giving helpful comments. He is grateful to Guram Bezhanishvili for helpful discussion concerning this work.
References


Graduate School of Information Science,  
Japan Advanced Institute of Science and Technology,  
Tatsunokuchi, Ishikawa, 923-1292, Japan  
e-mail: yasusi-h@jaist.ac.jp

( \textit{The present address} )  
Department of Management Information,  
Tokiwa Junior College,  
Mito, Ibaraki, 310-8585, Japan  
e-mail: hasimoto@tokiwa.ac.jp