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AN INTERNAL DETERMINACY METATHEOREM FOR ŁUKASIEWICZ'S *AUSSAGENKALKÜLS*

Abstract

An internal determinacy metatheorem is proved for Łukasiewicz's three-valued propositional calculus. The metatheorem establishes that no sentences of the logic are logically undetermined in truth value. The result is extended to Kleene's three-valued logic and related systems, but the significance of the metatheorem is shown by the fact that it does not apply indiscriminately to any and all three-valued logics. Implications of the metatheorem for five philosophical topics are indicated.

1. Łukasiewicz's Logic

Łukasiewicz's [1930] three-valued *Aussagenkalküls* allows propositions to have the values '1', '0','1/2' (see [9]). For convenience in comparison with standard truth-value semantics, these will be interpreted in what follows as 'true' (T), 'false' (F), and 'undetermined' (U)[1].

The exact axiomatization of Łukasiewicz's logic is unimportant for present purposes, but several versions of the theory have been offered (see [1], [2], [11]). Proof of the internal determinacy metatheorem requires only a consideration of the logic's characteristic nonstandard truth value semantics, which can be given as truth tables or matrix definitions by cases

[1] A similar convention is adopted by Nicholas Rescher. Rescher uses "I" for the third 'intermediate' or 'indeterminate' truth value in his exposition of Łukasiewicz's system, see [10], pp.22.

of some choice of primitive propositional connectives. Here negation and the conditional are defined, to which the other truth functions are reducible in the usual way.

P	$\sim P$
T	F
F	T
U	U

P	Q	$P \supset Q$
T	T	T
T	F	F
T	U	U
F	T	T
F	F	T
F	U	T
U	T	T
U	F	U
U	U	T

Where the antecedent is true and the consequent undetermined, and where the antecedent is undetermined and the consequent false, the conditional as a whole is undetermined. It may be tempting because of this to understand the third truth value epistemically, regarding a proposition's being undetermined as meaning it might be true or false, though it is not known which. Yet the epistemic reading is implausible for the final case, in which antecedent and consequent are both undetermined, but the conditional is true. If the epistemic rationale were consistently applied, it should otherwise be expected, as in Kleene's [1952] three-valued logic, explained below, that the conditional would also be undetermined. But since this is not Łukasiewicz's stipulation, it seems more appropriate to read his undetermined or '1/2' third truth value as a distinct value in its own right, rather than as disguising an unknown or unrevealed standard truth value. Łukasiewicz's logic in this intuitive sense is an ontic or realist rather than epistemic three-valued system[2].

2. Internal Determinacy Metatheorem

There is an easy proof that no proposition of Łukasiewicz's three-valued logic is logically undetermined. The proof is by mathematical induction on the length of its well-formed formulas (wffs).

[2]For background about Łukasiewicz's interpretation of and philosophical motivation for introducing the third value (see [6], [7], [8] and [9]).

Proof. Let L be Łukasiewicz's *Aussagenkalküls*, P a wff of L, and $V(P)$ the truth value of P. Then:

1. If $P = Q \supset R$, then by the truth table for the conditional '$\supset$', $V(P) = U$ if and only if either $V(Q) = T$ and $V(R) = U$ or $V(Q) = U$ and $V(R) = F$. But these conditions, according to the truth tables, are not logically guaranteed by the values of Q and R interrelated in the truth tables, unless Q or R is logically undetermined. Thus, P is not logically undetermined unless Q or R is logically undetermined.

2. If $P = \sim Q$, then by the truth table for negation '$\sim$', $V(P) = U$ if and only if $V(Q) = U$. But then P is not logically undetermined unless Q is logically undetermined.

3. If P is not truth functionally complex, then $V(P) = T, V(P) = F$, or $V(P) = U$. But then P is not logically undetermined.

Hence, no wff of L is logically undetermined. □

Although some propositions of Łukasiewicz's logic are logically true (e.g., $P \supset P, P \supset (P \supset P)$, etc.), and others are logically false (e.g., $\sim (P \supset P)$, etc.), none is logically undetermined. Łukasiewicz's three-valued semantics permit some, but do not require any, truth functionally complex propositions to be undetermined. The truth valuation of a proposition as undetermined can only come from outside the logic in an extralogical application, and introduced by the special content of substitution instances for propositional components of axioms or theorems. The logic considered in itself as a pure abstract system is internally determinate, in the sense that every truth functionally complex proposition of the system is determinately logically true, logically false, or logically contingent, and none logically undetermined. It remains for a consistency metatheorem to prove that there is a model or interpretation whereby every truth functionally complex proposition is logically true rather than logically false or logically contingent.

3. Extension to Kleene's Three-Valued Logic

The metatheorem holds also for Kleene's three-valued propositional semantics. Kleene's system shares the same truth table for negation as Łukasiewicz's, but offers in place the following alternative definition of the conditional:

P	Q	$P \supset Q$
T	T	T
T	F	F
T	U	U
F	T	T
F	F	T
F	U	T
U	T	T
U	F	U
U	U	U

As previously explained, the difference between Kleene's and Łukasiewicz's semantics is that Łukasiewicz evaluates a conditional with undetermined antecedent and consequent as true, while Kleene evaluates it as undetermined (see [3], pp. 334-335).

This, however, is an important difference, reflecting a distinction in attitude toward the contribution of the third truth value to truth functional compounds. It allows in Kleene's system a more naturally epistemic reading of the semantics of undetermined values, more suited for the purposes of an intuitionistic metamathematics[3].

The internal determinacy metatheorem applies to Kleene's three-valued logic with only the following revision of proof clause (1) for system K.

1*. If $P = Q \supset R$, then by the (Kleene) truth table for the conditional '$\supset$', $V(P) = U$ if and only if either $V(Q) = T$ and $V(R) = U, V(Q) = U$ and $V(R) = F$, or $V(Q) = V(R) = U$. But these conditions are not logically guaranteed by the values of Q and R, according to the truth tables, unless Q or R is logically undetermined. (Where $Q = R, V(P) = T$ or $V(P) = T$ or $V(P) = U$, depending on whether $V(Q) = V(R) = T$ or F, or $V(Q) = V(R) = U$, unless Q and R are logically undetermined.) Thus, P is not logically undetermined unless Q or R is logically undetermined.

(2) and (3), as above.

Hence, no wff of K is logically undetermined. □

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[3]See [4]. "In Kleene's system, a proposition is to bear the third value I not for fact-related, ontological reasons, but for knowledge-related epistemological ones: it is not to be excluded that the proposition may *in fact* be true or false, but it is morely *unknown* or undeterminable what its specific truth status is.", [10], p. 34.

4. Limitations of the Metatheorem

That the internal determinacy metatheorem holds for both Łukasiewicz's and Kleene's three-valued logics is noteworthy, because these arguably are the two theoretically most important, intuitively plausible, and useful, of all three-valued systems. The remaining variants either do not preserve analogues of the classical material conditional when projected into three values, or do not relate true antecedents and undetermined consequences or undetermined antecedents and false consequents in interesting ways.

Nevertheless, the metatheorem does not apply to all three-valued logics, and the fact that it can be shown not to obtain for certain of these relatively more deviant or nonstandard three-valued systems different from Łukasiewicz's and Kleene's demonstrates the metatheorem's significance, and provides further partial informal confirmation of the value of Łukasiewicz's and Kleene's logics.

It may be worthwhile first to recount why the metatheorem induction strategy is not so powerful as to be capable also of proving that no proposition of L is logically true or logically false. Consider what clause (1) establishes. The heart of the proof is the claim, supported by truth table definitions, that the conditions under which antecedent and consequent have the distribution of nonstandard truth values needed to make the conditional to which they belong undetermined in truth value are not logically guaranteed by their truth values unless they are already logically undetermined. This is just a fact about the truth table definitions as they are constituted, revealed by inspection. But the tables are also so defined that they do not permit corresponding proofs that no propositions of the logics are logically true, or that none is logically false. The proposition $P \supset P$ is logically true in L (though not in K), where the conditional is true, and hence a nonstandard tautology, regardless of the value of P. By contrast with the metatheorem's clauses (1) and (1*), there is a logical guarantee, at least in L, that the conditions required for a conditional proposition to be logically true by the values of its antecedent and consequent according to the truth tables, namely, among indeterminately many others, when antecedent and consequent are equivalent. Similarly, there is no parallel metatheorem to show that no proposition of these logics is logically false, a task for their consistency proofs, as follows at least for L, from the truth table for negation in the obvious counterexample, $\sim (P \supset P)$.

In Kleene's K, on the other hand, interestingly enough, there are no counterparts of classical tautologies or inconsistencies. Any construction out of the primitive syntax is logically guaranteed by the system's truth tables to be neither logically true, logically false, nor logically undetermined. This means that there are also forthcoming in Kleene's logic negative metatheorems to the effect that no proposition of K is logically true, and none logically false, corresponding to the internal determinacy metatheorem for L and K that none is logically undetermined. These collectively have the consequence that Kleene's three-valued logic is internally contingent as well as consistent, for the system considered as a pure abstract formalism prior to applications contains all and only trivalently logically contingent propositions. True and false propositions in Kleene's system occur along with undetermined propositions only in applications of the logic. In its intended application, these are found in the existence or nonexistence of extralogical mathematical proofs (or scientific verifications) or lack thereof, together with negation.

All three-valued logics are nonstandard, but some are more so than others. The following extraordinary three-valued truth table provides an illustration sufficient to show that the internal determinacy metatheorem does not hold for all three-valued logics, though it may indeed for this reason discriminate nicely between plausible and implausible systems.

P	Q	$P \supset Q$
T	T	T
T	F	F
T	U	U
F	T	T
F	F	U
F	U	T
U	T	T
U	F	U
U	U	U

Just as there are highly deviant, perhaps uninteresting, bivalent semantics for the conditional, according to which no conditional is a tautology, so there are deviant three-valued logics in which, contrary to the internal deteminacy metatheorem for Łukasiewicz's and Kleene's systems, some propositions are logically undetermined. Here is one obvious possibility, suggesting how other even more highly deviant three-valued semantics fall outside the internal determinacy metatheorem, for which the theorem does not obtain and cannot be proved.

The system in question is just like Kleene's, except for the case where antecedent and consequent are both F, in which the conditional is undetermined. Again, the semantics is implausible, failing to preserve classical truth conditions for the conditional in a three-valued environment. But it has the effect of logically guaranteeing that some propositions are undetermined, or that even internally the logic contains some logically undetermined propositions, such as, for example (with negation the same as always), $\sim (P \supset P) \supset \sim (P \supset P)$. Another logic with the same effect no further afield, but based on Łukasiewicz's semantics, imitates the table above in every respect until the final line, where undetermined antecedent and consequent are evaluated as producing a true rather than undetermined conditional. Other designer combinations can be contrived to guarantee logically or internally undetermined propositions.

It follows that the internal determinacy metatheorem says something significant about the limited range of three-valued logics to which it applies. For it does not hold true of all three-valued systems, but discriminates among them, applying to interesting and intuitively plausible versions, but not to others. The complete elaboration and proof of necessary and sufficient conditions for the metatheorem to obtain in a three-valued logic remains as a topic for continued research. But it appears obvious that the metatheorem minimally requires such distribution of truth values that no truth functionally complex proposition with the form of a classical tautology or its negation is undetermined according to the logic's truth table semantics. It is only under such conditions that truth functionally complex propositions of the logic will be logically guaranteed to be evaluated as undetermined regardless of the truth values of their propositional components.

5. Philosophical Implications

The internal determinacy metatheorem for Łukasiewicz, Kleene, and related multiple-valued logics has a number of philosophically important implications, of which I shall briefly discuss the following five.

(i) *Internal semantic isomorphism*

An exact semantic isomorphism with the inference structures of classical bivalent logic obtains for those and only those three-valued logics to which the internal determinacy metatheorem applies. By this means, the classical deduction theorem can be extended without modification to all such logics, considered internally or in themselves, as pure abstract formalisms, independently of their applications. This means that internally determinate three-valued logics behave semantically exactly like classical bivalent systems, unless or until propositions undetermined in truth value are introduced in extralogical applications.

(ii) *Simplification of metatheory*

This fact further implies that consistency, completeness, and compactness proofs in the metatheoredcal characterization of internally determinate three-valued logics can be highly simplified, reduced to those of classical bivalent logics with corresponding syntax. There is strictly no need, since the metatheory of such systems concerns their properties, not in application, but as pure abstract formalisms, for special provision to accommodate three values, because no propositions can have the third value, or any value beyond true or false, if the internal determinacy metatheorem applies to systems in which they occur. It follows that the elegant machinery used to consider undetermined or third-valued constructions in consistency, completeness, or compactness proofs of three-valued logic (mapping three values onto two, and the like) offered, for example, by Leblanc, Goldberg, and Weaver, is strictly unnecessary (see [5]).

(iii) *Criteria of distinction*

To the range of formal criteria for distinguishing different kinds of three-valued logics, the internal determinacy metatheorem adds another basis for discrimination into a philosophically interesting set of categories. There are internally determinate and internally nondeterminate three-valued logics. It happens that what have typically been regarded as the most plausible and useful three-valued logics are governed by the internal determinacy metatheorem. But since these logics are not internally three-valued, but only in application, they might finally come to be regarded as uninteresting *qua* three-valued logics. Attention may then shift to traditionally less favored logics that are three-valued not only in application, but also considered internally as pure abstract logics. These are systems that contain within themselves propositions that are logically undetermined, or third-valued by definition, and as such they are the counterparts of classical bivalent systems with logically true tautologies and logically false inconsistencies.

(iv) *Pure abstract v. applied logic*

The internal determinacy metatheorem highlights the received distinction between pure abstract and applied logical formalisms, between theory and practice in mathematical logic. The difference in pure and applied logic marked by the internal determinacy metatheorem, where their semantic properties are so dramatically different, warrants deeper philosophical reflection on the relation between pure and applied systems generally.

(v) *Gap and larger many-valued logics*

There is a direct extension of the internal determinacy metatheorem from Łukasiewicz and Kleene three-valued logics to correspondingly definable gap and larger many-valued logics. The adaptation to at least some gap logics is obvious enough. It requires only a uniform substitution of references to gaps for the third truth value throughout the metatheorem proof, for systems that have gaps where Lukasiewicz's or Kleene's semantics has the undetermined or third truth value. For larger many-valued systems, the only requirement is that their definitions of primitive propositional connections do not entail that truth functionally complex propositions with the syntactical structure of classical tautologies or their negations do not have any of the larger truth values when their component propositions are true or false, or when they have any of the other larger values. The extension of the internal determinacy metatheorem also extends the philosophical implications in (i)-(iv) to appropriate gap and larger many-valued logics.

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References

[1] Luisa Iturrioz, *An Axiom System for Three-Valued Łukasiewicz Propositional Calculus*, **Notre Dame Journal of Formal Logic** 18 (1977), pp. 616–620.

[2] Jacek K. Kabzinski, *On Equivalential Fragment of the Three-Valued Logic of Lukasiewicz*, **Bulletin of the Logic Section** 8 (1979), pp. 182–187 (Warsaw).

[3] S. C. Kleene, **Introduction to Metamathematics** (Amsterdam: North-Holland, 1952).

[4] S. C. Kleene, *On a Notation for Ordinal Numbers*, **The Journal of Symbolic Logic** 3 (1938), pp. 150–155.

[5] Hugues Leblanc, Harold Goldberg, George Weaver, *A Strong Completeness Theorem for Three-Valued Logic*, **Notre Dame Journal of Formal Logic** 15 (1974), pp. 325–330.

[6] J. Łukasiewicz, *Investigations into the semantical calculus*, [in:] **Selected Works**, L. Borkowski (ed.), Amsterdam, North-Holland, 1970, pp. 131-153.

[7] J. Łukasiewicz, *On the principle of the excluded middle*, trans. by J. Woleński and P. Simons, **History and Philosophy Logic** 8 (1987), pp. 67-69.

[8] J. Łukasiewicz, *On three-valued logic*, [in:] **Selected Works**, L. Borkowski (ed.), Amsterdam, North-Holland, 1970, pp. 87-88.

[9] J. Łukasiewicz, *Philosophical remarks on many-valued systems of propositional logic*, [in:] **Selected Works**, L. Borkowski (ed.), Amsterdam, North-Holland, 1970, pp. 153-178.

[10] N. Rescher, **Many-valued logic**, New York, McGraw-Hill, 1969.

[11] Mordchaj Wajsberg, *Axiomatization of the Three-Valued Propositional Calculus [1931]*, in Polish Logic: 1920-1939, Storrs McCall, ed. (Oxford: Clarendon Press, 1967), pp. 264-284.

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