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AN INFINITARY EXTENSION OF $MALL^{-}$

Abstract

The aim of the present paper is to introduce an extension of the Multiplicative Fragment of Classical Propositional Linear Logic where infinitary versions of $\otimes$ and ⅋ are considered. We define the sequent calculus LIP_∞ and prove its soundness and completeness with respect to an infinitary extension of Girard's Phase Semantics.

1. Introduction

Infinitary propositional systems for classical and intuitionistic logic have been studied in several works [5, 4]. The aim of the present paper is to introduce an extension of the Multiplicative Additive Fragment of Classical Propositional Linear Logic (without ! and ?) ($MALL^{-}$) where infinitary versions of $\otimes$ and ⅋ are considered. We will define an infinitary sequent calculus, LIP_∞, and prove its completeness and soundness with respect to an extension fo Girard's Phase Semantics.

The paper is organized in the following way: in Section 2 we introduce the infinitary system LIP_∞, in Section 3 we define the infinitary Phase Semantics and in Section 4 we prove the soundness and the completeness of the system introduced in Section 2.

2. The system LIP$_\infty$

The language of LIP$_\infty$ is the same as that of the Multiplicative Fragment of Classical Linear Propositional Logic augmented by $⅋_{i<\omega}$ and $\otimes_{i<\omega}$. We add to the usual definition of formula the following clauses: if $(A_i)_{i<\omega}$ is a set of LIP$_\infty$ formulas, then $⅋_{i<\omega}(A_i)$ and $\otimes_{i<\omega}(A_i)$ are LIP$_\infty$ formulas. The linear negation of these propositions are defined as $\otimes_{i<\omega}(A_i^\perp)$ and $⅋_{i<\omega}(A_i^\perp)$, respectively. Below we present the sequent calculus (LIP_∞). A sequent is a pair of multisets (possibly infinite multisets) of LIP$_\infty$ formulas. We shall adopt throughout the paper one-sided sequents, as in [2]. A sequent is, thus, an expression of the $\Rightarrow \Gamma$, where Γ is a (possibly infinite) multiset of formulas. We remind that the union of multisets is applicable to infinite multisets.

The natural infinitary generalization of the tensor operator does not introduce any significant additional difficulty. On the other hand, the natural infinitary generalization of the par operator does not seem to allow for a natural (proof-theoretical) concept of reduction. In a certain sense, as it was remarked by Blass [1], our usual concept of sequent and our usual concept of derivation as a well–founded tree are not adequate for the definition of a ⅋ reduction. In this paper we do not solve this well-foundedness problem. But, we do show that the system with an infinitary cut is sound and complete. We think that the introduction of infinitary cuts is an important step towards an adequate proof-theoretical approach to LIP_∞.

$$\frac{}{\Rightarrow A, A^\perp}\ \text{Id Axiom} \qquad \frac{}{\Rightarrow \top, \Delta}\ \text{Axiom}-\top \qquad \frac{}{\Rightarrow 1}\ \text{Unity}$$

$$\frac{(\Rightarrow A_i, \Delta_i)_{i<\omega} \qquad \Rightarrow (A_i^\perp)_{i<\omega}, \Gamma}{\Rightarrow (\Delta_i)_{i<\omega}, \Gamma}\ \infty\text{-Cut} \qquad \frac{\Rightarrow A, \Delta \qquad \Rightarrow A^\perp, \Gamma}{\Rightarrow \Delta, \Gamma}\ \text{Cut}$$

$$\frac{\Rightarrow \Delta}{\Rightarrow \Delta, \perp}\ \perp\text{-Rule} \qquad \frac{\Rightarrow A, \Delta \qquad \Rightarrow B, \Delta}{\Rightarrow A\&B, \Delta}\ \&\text{-Rule}$$

$$\frac{\Rightarrow A, \Delta}{\Rightarrow A \oplus B, \Delta}\ \oplus_1\text{-Rule} \qquad \frac{\Rightarrow B, \Delta}{\Rightarrow A \oplus B, \Delta}\ \oplus_2\text{-Rule}$$

$$\frac{\Rightarrow A, \Delta \qquad \Rightarrow B, \Gamma}{\Rightarrow A \otimes B, \Delta, \Gamma}\ \otimes\text{-Rule} \qquad \frac{(\Rightarrow A_i, \Delta_i)_{i<\omega}}{\Rightarrow \otimes_{i<\omega}(A_i), (\Delta_i)_{i<\omega}}\ \infty\text{-}\otimes\text{-Rule}$$

$$\frac{\Rightarrow A, B, \Delta}{\Rightarrow A ⅋ B, \Delta}\ ⅋\text{-Rule} \qquad \frac{\Rightarrow (A_i)_{i<\omega}, \Delta}{\Rightarrow ⅋_{i<\omega}(A_i), \Delta}\ \infty - ⅋\text{-Rule}$$

The system LIP_∞ is given by the axioms and rules of inference in the above box.

3. Algebraic Semantics

Infinitary Phase Spaces (PS_∞) are a straightforward extension of the usual Phase space to cope with the infinitary features of LIP_∞. In the sequel we present PS_∞, taking for granted some basic facts about Phase Space semantics.

Instead of using an ordinary monoid, as Girard does, we use a monoid extended with an infinitary operation compatible with the original monoid operation. In what follows, we denote by M^ω, where M is a set, the set of denumerably infinite sequences of elements of M. This notation can be formally taken as the set of functions $f : \mathsf{N} \longrightarrow M$. Thus we have the following definition:

DEFINITION 3.1. An ∞-Monoid $\mathcal{M}$ is a structure $< M, \star, \overset{\infty}{\star}_{i<\omega}, 1 >$, where $< M, \star, 1 >$ is a commutative monoid and $\overset{\infty}{\star}_{i<\omega}$ is an infinitary operation on M, that is, a function from M^ω into M such that:

- $\overset{\infty}{\star}_{i<\omega} (a_i) = \overset{\infty}{\star}_{i<\omega} (a_{p(i)})$, where $p : \mathsf{N} \longrightarrow \mathsf{N}$ is any bijection (onto and 1-1 function).
- Let $c_{2i} = a_i$ and $c_{2i+1} = b_i$, then $\overset{\infty}{\star}_{i<\omega} (a_i) \star \overset{\infty}{\star}_{i<\omega} (b_i) = \overset{\infty}{\star}_{i<\omega} (c_i)$.
- Let $a_o = b$ and $a_i = b_{i-1}$, for $i > 0$, then $\overset{\infty}{\star}_{i<\omega} (a_i) = b \star \overset{\infty}{\star}_{i<\omega} (b_i)$.

We use the more liberal notation $\overset{\infty}{\star}_{i<\omega} (a_i)$ simply to indicate the functional nature of the infinitary operation.

DEFINITION 3.2. An infinitary Phase Space, PS_∞ for short, consists of a ∞-monoid, denoted by P, and a subset $\perp_P \subseteq P$, the so called set of antiphases of P.

The algebraic definitions found in [2], of the operation "$\multimap$" and of *Facts* remain unchanged. In what follows we introduce our algebraic approach to the new infinitary connectives together with the original definitions. We shall be mainly concerned with the multiplicative connectives, since the additives remain unchanged.

Definition 3.3. Let $\mathcal{P}$ be a PS_∞. Then, we can define the following operations on the subsets of elements of the underlying set of PS_∞. Such subsets will be denoted by G and H (possibly with indexes).

- $G \parr H = (G^\perp \star H^\perp)^\perp$
- $G \otimes H = (G \star H)^{\perp\perp}$
- $G \multimap H = (G \star H^\perp)^\perp$
- $\parr_{i<\omega}(G_i) = (\overset{\infty}{\star}_{i<\omega} (G_i^\perp))^\perp$
- $\otimes_{i<\omega}(G_i) = (\overset{\infty}{\star}_{i<\omega} (G_i))^{\perp\perp}$

We recall that a subset H of the underlying set of a PS_∞ is a fact iff $H = H^{\perp\perp}$. There is an equivalent definition, in fact a property, of fact, namely, *H is a fact iff $H = G^\perp$ for some $G \subseteq PS_\infty$* (cf. [2]).

We can easily verify that $(\otimes_{i<\omega}(G_i^\perp))^\perp = ((\overset{\infty}{\star}_{i<\omega} G_i^\perp)^{\perp\perp})^\perp$ and that the right-hand side of this equation can also be read as $((\overset{\infty}{\star}_{i<\omega} G_i^\perp)^\perp)^{\perp\perp}$. By the definition we have that $(\overset{\infty}{\star}_{i<\omega} G_i^\perp)^\perp$ is a fact, and hence, by involution, we have that $\otimes_{i<\omega}(G_i^\perp)^\perp = (\overset{\infty}{\star}_{i<\omega} G_i^\perp)^\perp$, which is, by definition, $\parr_{i<\omega} G_i$. Finally, we have that $(\otimes_{i<\omega}(G_i^\perp))^\perp = \parr_{i<\omega} G_i$.

We can easily obtain from the definitions above (particularly from the third item in definition refinfmonoid) the following:

Fact 3.4. Let P be a PS_∞ and $(G_i)_{i<\omega} \subseteq P$ be facts. Then $\parr_{i<\omega}(G_i) = G_0 \parr \parr_{0<i<\omega}(G_i)$.

In the same spirit, from the second item of the Definition 3.1 we can conclude that:

Fact 3.5. Let $(H_i)_{i<\omega}$ and $(G_i)_{i<\omega}$ be families of facts under the same PS_∞. Consider $(W_i)_{i<\omega}$ a sequence formed only with members of $(H_i)_{i<\omega}$ and $(G_i)_{i<\omega}$ such that the internal order within the respective families are preserved and the $G's$ appear after the $H's$[1]. Then, we have that $\parr_{i<\omega}(W_i) = \parr_{i<\omega}(H_i) \parr \parr_{i<\omega}(G_i)$.

Let H and G be facts. We can note that $1 \in H \multimap G$ iff $H \subseteq G$. Another useful fact is the following:

[1] In set theoretical terms we could say the W_i is of order type $\omega + \omega$ whenever G_i and H_i are of order type ω respectively

FACT 3.6. Let H_j be a fact for each $j \in \mathsf{N}$, and Δ_j be an denumerably infinite family of facts for each $j \in \mathsf{N}$. Then, for each $j \in \mathsf{N}$, $\{H_j\} \cup \Delta_j$ is an infinite and denumerable family of facts. Under those conditions we have that if $1 \in \parr_{i<\omega}(\{H_j\} \cup \Delta_i)$ for each j, then $1 \in \otimes_{i<\omega}(H_i) \parr \parr_{i<\omega}(\Theta_i)$, where Θ_i is any enumeration of $\bigcup_{i<\omega} \Delta_i$.

PROOF. From, $1 \in \parr_{i<\omega}(\{H_j\} \cup \Delta_i)$ we have, by Fact 3.4, that $1 \in H_j \parr \parr_{i<\omega}(\Delta_i)$. By the definition of $\multimap$, we obtain $1 \in (\parr_{i<\omega}(\Delta_i))^\perp \multimap H_j$. Thus, $(\parr_{i<\omega}(\Delta_i))^\perp \subseteq H_j$ for each j, and,

$$\otimes_{j<\omega}(\parr_{i<\omega}(\Delta_i))^\perp \subseteq \otimes_{j<\omega}(H_j)$$

From this it immediately follows that:

$$1 \in \otimes_{j<\omega}(\parr_{i<\omega}(\Delta_i))^\perp \multimap \otimes_{j<\omega}(H_j)$$

which in turn implies that

$$1 \in (\otimes_{j<\omega}(H_j)) \parr (\otimes_{j<\omega}(\parr_{i<\omega}(\Delta_i))^\perp)^\perp.$$

Finally, we can show that

$$1 \in (\otimes_{j<\omega}(H_j)) \parr (\parr_{j<\omega}(\parr_{i<\omega}(\Delta_i))),$$

and, by a suitable choice of an enumeration for $\bigcup_{i<\omega} \Delta_i$, we reach the desired conclusion.

As in [2], a Linear Structure for a Propositional Language $\mathcal{L}$ is a pair $< \mathcal{P}, \mathcal{S} >$, where $\mathcal{P}$ is a PS_∞, and S is a mapping that assigns to each propositional letter A a fact $S(A)$ in $\mathcal{P}$. This mapping is extended in a straightforward way to every LIP_∞ formula. Thus, each formula α is interpreted as a fact $S(\alpha)$. We say that α is valid in $\mathcal{P}$ iff $1 \in S(\alpha)$. A formula is a LIP_∞ tautology iff it is valid in every PS_∞ structure.

DEFINITION 3.7. Let $\mathcal{L}$ be a language and $< \mathcal{P}, \mathcal{S} >$ a Linear Structure for it. We say that a LIP_∞ sequent $\Rightarrow \Delta$ is valid with regard to this linear structure iff:

- $\Delta = \{\delta_0, \ldots, \delta_n\}$ and $1 \in S(\delta_0 \parr \delta_1 \ldots \parr \delta_n)$; or
- $\Delta = \{\delta_0, \ldots, \delta_n, \ldots\}$ is an infinite multiset and $1 \in S(\parr_{i<\omega}(\delta_i))$

4. Soundness and Completeness of LIP_∞

In this section we shall prove that LIP_∞ is sound and complete w.r.t the LIP_∞ semantics described in Section 3.

THEOREM 4.1. *LIP_∞ is sound regarded to (infinitary) Linear Structures.*

PROOF. It suffices to show that the axioms are valid and the rules preserve validity. In what follows we shall use the notation $\parr_{i<\omega}(\Delta)$ to indicate that the infinitary operator is applied to the elements of Δ, without explicitily mention to the order in which they occur[2]. We will also consider $S(\Delta)$ as an indexed set, with members $S(\delta)$ for each $\delta \in \Delta$.

- Id Axiom and Unity axioms inherite their validity from ordinary phase spaces.
- Axiom $\top$, in cases where $\Delta = \{\delta_0, \ldots, \delta_n, \ldots\}$ is infinite, has to be verified. Using Fact 3.4, we have that its interpretation is of the form $S(\top) \parr \parr_{i<\omega}(\delta_i)$, with $\parr_{i<\omega}(\delta_i)$ being a fact. From this, it follows, as it is seen in [2], that $1 \in S(\Rightarrow \top, \Delta)$.
- Let us consider the &-Rule. Suppose that the premisses are valid, i.e, $1 \in \parr_{i<\omega}(\{S(A)\} \cup S(\Delta))$ and $1 \in \parr_{i<\omega}(\{S(B)\} \cup S(\Delta))$. By applying Fact 3.4 to both premises we have that $1 \in S(A) \parr \parr_{i<\omega}(S(\Delta))$ and that $1 \in S(B) \parr \parr_{i<\omega}(S(\Delta))$. Since $\parr_{i<\omega}(S(\Delta))$ is a fact, we can use Girard's original argument for ordinary finitary case ([2]) and conclude that $1 \in (S(A) \& S(B)) \parr \parr_{i<\omega}(S(\Delta))$. By means of a second application of Fact 3.4 we infer that $1 \in \parr_{i<\omega}(\{(S(A) \& S(B))\} \cup S(\Delta)))$. From this it immediately follows the validity of the $\Rightarrow A \& B, \Delta$.
- The justification for the rules Cut, $\perp$-Rule, $\otimes_1$-Rule, $\otimes_2$-Rule, $\otimes$-Rule and $\parr$-Rule is similar to that for the item discussed above (&-Rule).
- In the case of $\infty - \parr$-rule, we just have to observe that soundness is a direct consequence of the interpretation of an infinite sequents together with Fact 3.5.
- The soundness of the $\infty - \otimes$-rule follows directly from Fact 3.6.
- The soundness of the ∞-cut is obtained in the following way. We first observe that $1 \in \parr_{j<\omega}(\{S(A_i)\} \cup S(\Delta_i))$, for each i, implies that $1 \in \otimes_{i<\omega}(S(A_i)) \parr \parr_{i<\omega}(S(\Delta_i))$ (Fact 3.6). From the hypothesis applied to other premiss we also know that $1 \in \parr_{i<\omega}(S(A_i)^\perp) \parr \parr_{i<\omega}(\Gamma)$.

[2] We know that the ordering is not essential by the first item of Definition 3.1

Thus, given that $\otimes_{i<\omega}(S(A_i)) = (⅋_{i<\omega}(S(A_i)^\perp))^\perp$, we can apply the same analysis used in the case finitary cuts ([2]) to obtain the desired conclusion.

THEOREM 4.2. *Every tautology is provable in CS_∞.*

PROOF. The proof proceeds as in [2]. In the original proof, Girard used the the (commutative) monoid of finite Multisets of formulas with the empty multiset as the neutral element and the union of multisets as the monoid operation. Then, he introduced a Phase Structure whose set of anti-phases was defined as the set of provable sequents. Finally, he considered the mapping $Pr(\alpha) = \{\beta / \vdash \alpha, \beta \text{ is provable }\}$ and showed that it induces an interpretation over this Phase Space. If a formula α is a tautology, then it is valid w.r.t. this Phase Space. By definition, this means that the empty multiset (the neutral element of the monoid) belongs to $Pr(\alpha)$. From this we immediately conclude that α is provable. The proof that this mapping really induces an interpretation uses the property $Pr(\alpha^\perp) = Pr(\alpha)^\perp$ proved in [2]. The proof of this property, when carried out in our context, remains unchanged.

In the case of LIP_∞, the elements of the monoid will be finite and infinite Multisets of formulas. The Monoid operation is the Multiset union, and the empty Multiset is again the neutral element. Notice that the ordinary monoid operation ($\star$), as well its infinitary version ($\overset{\infty}{\star}_{i<\omega}$), are represented by the same set-theoretical operation. However, for the sake of notational convenience, we shall use the symbols $\star$ and $\overset{\infty}{\star}_{i<\omega}$ to indicate it. We can easily see that the structure just described is a commutative ∞-monoid. The set of anti-phases is the set of LIP_∞ provable sequents (viewed as multisets). The mapping Pr is defined in the finitary case. In order to establish the final argument of the proof, we need to show that Pr induces an interpretation in the PS_∞ considered above. That is, we need to show that

- $Pr(\alpha)$ is a fact for every formula α
- $Pr(\alpha \otimes \beta) = Pr(\alpha) \otimes Pr(\beta)$
- $Pr(\alpha \parr \beta) = Pr(\alpha) \parr Pr(\beta)$
- $Pr(\alpha \multimap \beta) = Pr(\alpha) \multimap Pr(\beta)$
- $Pr(\alpha \& \beta) = Pr(\alpha) \& Pr(\beta)$
- $Pr(\alpha \oplus \beta) = Pr(\alpha) \oplus Pr(\beta)$
- $Pr(\parr_{i<\omega}(\alpha_i)) = \parr_{i<\omega}(Pr(\alpha_i))$
- $Pr(\otimes_{i<\omega}(\alpha_i)) = \otimes_{i<\omega}(Pr(\alpha_i))$

The proof of each item, except the last two, can be found in [2]. Although we are using PS_∞'s instead of ordinary Phase Space, the proofs of the remaining items are completely analogous to their finitary counterparts. We show below in detail the proofs of the last two items.

- By the $\infty - \otimes$-rule, we have that $\overset{\infty}{\star}_{i<\omega}\ (Pr(\alpha_i)) \subseteq Pr(\otimes_{i<\omega}(\alpha_i))$. Given that $Pr(\otimes_{i<\omega}(\alpha_i))$ is a fact, it follows from the definition of $\otimes_{i<\omega}$ that $\otimes_{i<\omega}(Pr(\alpha_i)) \subseteq Pr(\otimes_{i<\omega}(\alpha_i))$. On the other hand, let us suppose that $\Delta \in Pr(\otimes_{i<\omega}(\alpha_i))$ ($\Rightarrow \Delta, \otimes_{i<\omega}(\alpha_i)$ is provable) and that $\Gamma \in (\overset{\infty}{\star}_{i<\omega}\ Pr(\alpha_i))^\perp$. By the definition of the set of anti-phases and the definition of Pr, we have that $\Rightarrow \Gamma, (\alpha_i^\perp)_{i<\omega}$ is provable, and hence, by an application of the ∞-$\parr$-rule, we have that $\vdash \Gamma, \parr_{i<\omega}(\alpha_i^\perp)$ is provable too. Thus, by an application of the cut, we infer that $\Rightarrow \Gamma, \Delta$ is provable, and hence $\Delta \in (\overset{\infty}{\star}_{i<\omega}\ Pr(\alpha_i))^{\perp\perp} = \otimes_{i<\omega}(Pr(\alpha_i))$.
- Given that $Pr(\parr_{i<\omega}\alpha_i) = Pr((\otimes_{i<\omega}\alpha_i^\perp)^\perp)$, using the proof of the previous case, we infer that $Pr((\otimes_{i<\omega}\alpha_i^\perp)^\perp) = (\otimes_{i<\omega}(Pr(\alpha_i^\perp)))^\perp$, and hence equals $\otimes_{i<\omega}(Pr(\alpha_i)^\perp)^\perp$. Using the infinitary version of the DeMorgan law, we reach the desired conclusion.

This concludes the proof of the desired result.

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