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A NORMALIZATION THEOREM FOR *P-W*

Abstract

In this note, we consider sequent calculus for an implicational non-commutative intuitionistic logic P-W, or equivalently called $BB'I$. We shall prove the *Normalization theorem for* $L_{BB'I}$, which Komori used in his syntactic proof of Martin's theorem, known as a solution to the $P - W$ problem.

1. Introduction

In this note, we consider an implicational non-commutative intuitionistic logic P-W or equivalently called $BB'I$, whose axioms are

(B) $(b \to c) \to ((a \to b) \to (a \to c))$,
(B') $(a \to b) \to ((b \to c) \to (a \to c))$,
(I) $a \to a$,

with substitution and modus ponens as rules.

The P-W problem is a problem asking the truth of the following statement:

- If $\alpha \to \beta$ and $\beta \to \alpha$ are provable in P-W, then $\alpha = \beta$.

Although the system P-W is extremely simple, the P-W problem turned out to be very challenging, and had left open for more than twenty years since Belnap had originally asked it (p. 95, [1]), until E.P. Martin

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solved it affirmatively [5]. Martin solved the problem by showing the statement below known as Powers' conjecture: Powers showed its equivalence to the P-W problem [6], essentially by obtaining the theory of combinators generated from B, B' and I.

- No formula of the form $\alpha \to \alpha$ is provable in P-W without using the axiom I.

Since Martin's solution was a semantical proof, finding a syntactic proof of the problem attracted some researchers. The first syntactic proof in Hilbert-style formal system was obtained by Kron [4]. The first syntactic proof in sequent calculus, called $L_{BB'I}$, was outlined by the author in 1994, and later improved and published by Komori in 1996 [3]. While the author showed the original statement for the P-W problem directly, which in fact resulted in a lot of complications in the proof, Komori successfully proved the statement of Powers in a simpler manner. Since Komori quoted in his proof a lemma due to the author called a *Normalization theorem for* $L_{BB'I}$ (p. 412, [3]), without proof, we shall prove it in this note. Let α be a formula with the rightmost variable p in the following form:

$$\alpha = \alpha_1 \to (\alpha_2 \to (\cdots \to (\alpha_{n-1} \to (\alpha_n \to p))\cdots)).$$

The Normalization theorem for $L_{BB'I}$ states, for a given provable sequent $\Delta, \alpha, \Gamma \Rightarrow \gamma$, that there exists a canonical cut-free proof whose initial part consists of n successive applications of the rule of the left introduction of implication for each formula α_i $(1 \leq i \leq n)$. This is a consequence derived in P-W from the permutability between the logical rules involved with α_i's and those not involved with.

Finally, we note that the first syntactic proof of Martin's theorem, based on simply typed λ-calculus, is proved in a forthcoming paper by Hirokawa, Komori and the author [2].

2. Normalization theorem for BB'I

Notation. Let $\Gamma, \Delta, \Sigma ...$ be sequences of formulas in sequents. As usual, a *merge* of the sequences Γ and Δ is a new sequence consisting of the members of Γ and Δ as multisets, in which both Γ and Δ preserve their original orders. A *guarded merge* of Γ and Δ, denoted by $\Gamma \circ \Delta$, is one of the merges obtained from Γ and Δ, in which the rightmost formula is the rightmost formula of Δ.

DEFINITION 2.1. We define a system $L_{BB'I}$ as follows:

Axioms: $p \Rightarrow p$, where p is a propositional variable.

Rules of inference:

$$\frac{\Gamma \Rightarrow \alpha \quad \Delta, \alpha, \Sigma \Rightarrow \gamma}{\Delta \circ \Gamma, \Sigma \Rightarrow \gamma} \text{ (cut)}$$

where $\Gamma \neq \emptyset$ or $\Delta = \emptyset$,

$$\frac{\Gamma, \alpha \Rightarrow \beta}{\Gamma \Rightarrow \alpha \rightarrow \beta} \text{ } (\rightarrow \text{right})$$

$$\frac{\Sigma \Rightarrow \alpha \quad \Delta, \beta, \Gamma \Rightarrow \gamma}{\Delta \circ (\{\alpha \rightarrow \beta\} \circ \Sigma), \Gamma \Rightarrow \gamma} \text{ } (\rightarrow \text{left})$$

where $\Gamma \neq \emptyset$.

We note that the cut-elimination theorem holds for $L_{BB'I}$ (p. 62, [1] and p. 410, [3]).

NOTATION. We denote by $\overline{\alpha_i}$ a subformula

$$\alpha_i \rightarrow (\cdots \rightarrow (\alpha_{n-1} \rightarrow (\alpha_n \rightarrow p)) \cdots) \ \ (1 \leq i \leq n).$$

Now we prove the main theorem in this note.

THEOREM 2.2. *(Normalization theorem for $L_{BB'I}$.) Let α be a formula with the rightmost variable p in the following form:*

$$\alpha = \alpha_1 \rightarrow (\alpha_2 \rightarrow (\cdots \rightarrow (\alpha_{n-1} \rightarrow (\alpha_n \rightarrow p)) \cdots)).$$

For a given sequent $\Delta, \alpha, \Gamma \Rightarrow \gamma$ provable in $L_{BB'I}$, there exists a cut-free proof of the sequent such that:

$$
\cfrac{\cfrac{\begin{matrix}\vdots \\ \Sigma_1 \Rightarrow \alpha_1\end{matrix} \qquad \cfrac{\cfrac{\begin{matrix}\vdots \\ \Sigma_i \Rightarrow \alpha_i\end{matrix} \quad \cfrac{\begin{matrix}\vdots & \vdots \\ \Sigma_n \Rightarrow \alpha_n & \Delta_n, p, \Gamma_n \Rightarrow p\end{matrix}}{\begin{matrix}\vdots \\ \Delta_i, \overline{\alpha_{i+1}}, \gamma_i \Rightarrow p\end{matrix}}}{\Delta_{i-1}, \overline{\alpha_i}, \Gamma_{i-1} \Rightarrow p}\,(\rightarrow left)}{\begin{matrix}\vdots \\ \Delta_1, \overline{\alpha_2}, \Gamma_1 \Rightarrow p\end{matrix}}}{\Delta_0, \alpha, \Gamma_0 \Rightarrow p}\,(\rightarrow left)}{\begin{matrix}\vdots \\ \Delta, \alpha, \Gamma \Rightarrow \gamma,\end{matrix}}
$$

where

(1) A part of the proof called as a tail:

$$
\begin{matrix} \Delta_0, \alpha, \Gamma_0 \Rightarrow p \\ \vdots \\ \Delta, \alpha, \Gamma \Rightarrow \gamma \end{matrix}
$$

does not contain any inference rule in which α is principal.

(2) $\Delta_{i-1}, \overline{\alpha_i}, \Gamma_{i-1} \equiv \Delta_i \circ (\{\overline{\alpha_i}\} \circ \Sigma_i), \Gamma_i \quad (1 \leq i \leq m)$.

Proof. We call the proof of $\Delta, \alpha, \Gamma \Rightarrow \gamma$ above a *normalized proof with respect to α*. We make an induction on the length of the cut-free proof of $\Delta, \alpha, \Gamma \Rightarrow \gamma$. When the length is 1, the sequent is an axiom, and the claim clearly holds. For the induction step, we argue according to the last inference of the cut-free proof of $\Delta, \alpha, \Gamma \Rightarrow \gamma$. The only non-trivial case is when the last inference is the following (remember that γ is a variable):

$$
\cfrac{\begin{matrix}\vdots \\ \Sigma_1 \Rightarrow \alpha_1\end{matrix} \quad \begin{matrix}\vdots \\ \Delta', \overline{\alpha_2}, \Gamma' \Rightarrow \gamma\end{matrix}}{\Delta' \circ \{\alpha\} \circ \Sigma_1, \Gamma' \Rightarrow \gamma}(\rightarrow \text{left}),
$$

where $\Delta, \alpha, \Gamma \equiv \Delta' \circ (\{\alpha\} \circ \Sigma_1), \Gamma'$.

By the induction hypothesis applied to $\Delta', \overline{\alpha_2}, \Gamma' \Rightarrow \gamma$, we have a transformation and its normalized proof with respect to $\overline{\alpha_2}$

$$\dfrac{\begin{array}{cc}\vdots & \vdots \\ \Sigma_2 \Rightarrow \alpha_2 & \Delta_2, \overline{\alpha_3}, \Gamma_2 \Rightarrow p\end{array}}{\Delta_1, \overline{\alpha_2}, \Gamma_1 \Rightarrow p}$$
$$\vdots$$
$$\Delta', \overline{\alpha_2}, \Gamma' \Rightarrow \gamma$$

with a tail we call P_1 of the following form:

$$\Delta_1, \overline{\alpha_2}, \Gamma_1 \Rightarrow p$$
$$\vdots$$
$$\Delta', \overline{\alpha_2}, \Gamma' \Rightarrow \gamma.$$

Now we first derive $\Delta_1 \circ (\{\alpha\} \circ \Sigma_1), \Gamma_1 \Rightarrow p$ by a proof

$$\dfrac{\begin{array}{cc}\vdots & \vdots \\ \Sigma_1 \Rightarrow \alpha_1 & \Delta_1, \overline{\alpha_2}, \Gamma_1 \Rightarrow p\end{array}}{\Delta_1 \circ (\{\alpha\} \circ \Sigma_1), \Gamma_1 \Rightarrow p}.$$

By applying the inference rules in P_1 to $\Delta_1 \circ (\{\alpha\} \circ \Sigma_1), \Gamma_1 \Rightarrow p$, we obtain $\Delta' \circ (\{\alpha\} \circ \Sigma_1), \Gamma' \Rightarrow \gamma$, which is equal to $\Delta, \alpha, \Gamma \Rightarrow \gamma$. Hence the claim follows.

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