NEW SINGLE AXIOMS FOR POSITIVE IMPLICATION

The first single axiom for the implicational fragment of the intuitionistic sentential calculus was discovered by C. A. Meredith who, in [1], uses substitution and detachment to derive the Tarski-Bernays base \( CCpCqrCCpqCpr \) and \( CpCqp \) in nineteen steps from

\[ M_1 \cdot CCCpqrCsCCqCrtCqt \]

Thomas [2] later reduced the number of detachments by two.

Write \( X \) for \( CCCpqr \), \( Y \) for \( s \), and \( Z \) for \( CqCrt \) so that \( M_1 \) is \( CXCYCZCqt \), and consider the permutations of \( M_1 \) that result from interchanging these in pairs:

\begin{align*}
\text{M2}. & \ CXCYCXCqt = CCqCrtCsCCCpqrCqt \\
\text{M3}. & \ CYCZXCqt = CqCrtCCCpqrCCqCrtCsCqt \\
\text{M4}. & \ CXCZCYCqt = CCCpqrCCqCrtCsCqt \\
\text{M5}. & \ CZXCYCqt = CCqCrtCCCpqrCcCqt \\
\text{M6}. & \ CYCXZCqt = CsCCCpqrCCqCrtCqt.
\end{align*}

THEOREM. Each of M1-M6 is a single axiom for positive implication.

For proof, we derive each from its predecessor, and finally complete the circle by deriving \( M_1 \) from \( M_6 \). Proofs are abbreviated by writing \( Dm.n \) for the most general result of detaching formula \( n \), or a substitution instance of it, from formula \( m \), or a substitution instance of it.
The derivation from $M_1$ of $M_2$ is quite short:

\[ M_1 = 1. \quad CCCpqrCsCCqCrtCqt \]
\[ D1.1 = 2. \quad CpCCqCCrCCsCqCstuCqu \]
\[ D2.1 = 3. \quad CCpCCqCCrCpCrsCrtCpt \]
\[ D3.1 = 4. \quad CCCpqrCCqCrsCqs \]
\[ D3.2 = 5. \quad CCpCqrCqCstCpsCr \]
\[ D5.4 = M_2. \quad CCqCrtCstCCCpqrCqt \]

And the route from $M_2$ to $M_3$ is one step shorter:

\[ M_2 = 1. \quad CCqCrtCstCCCpqrCqt \]
\[ D1.1 = 2. \quad CpCCCqCrCstuCCrCstCCCrsCrt \]
\[ D1.2 = 3. \quad CpCCCqCrCstCqCCrCstuCtv \]
\[ D3.1 = 4. \quad CCCpqCCrCsCtvCqCCsCstuCCCustCsu \]
\[ D4.2 = M_3. \quad CsCCqCrtCCCpqrCqt \]

A bit more involved is the derivation from $M_3$ of $M_4$:

\[ M_3 = 1. \quad CsCCqCrtCCCpqrCqt \]
\[ D1.1 = 2. \quad CCpCqrCCCpqCpr \]
\[ D2.1 = 3. \quad CCCpqCrCstCqCCrCstuCrt \]
\[ D3.2 = 4. \quad CCCpqCrrCstCpqCstpq \]
\[ D3.3 = 5. \quad CCCpqCrCstCCqCptCpr \]
\[ D5.5 = 6. \quad CCCpqCrCstCtvCqCCsCstuCCCustCsu \]
\[ D6.4 = M_4. \quad CCCpqCrCrtCstCqt \]

The path from $M_4$ to $M_5$ is more complicated:

\[ M_4 = 1. \quad CCCpqCrCrtCstCqt \]
\[ D1.1 = 2. \quad CCpCCCqCrCstCqCqrtCpu \]
\[ D1.2 = 3. \quad CCCCCqCprCcCqCtvCcCCCpCqrCstCtv \]
\[ D2.2 = 4. \quad CpCCCqCrCstCrsCstuCqu \]
\[ D4.1 = 5. \quad CCpCCCqCprCstCqrtCpt \]
\[ D5.4 = 6. \quad CpCCCqCprCqr \]
\[ D6.1 = 7. \quad CCCCCqCrsCCrCstuCrtCtvCpv \]
\[ D7.6 = M_5. \quad CCqCrtCCCpqrCstCqt \]
Lengthier yet is the deduction from \( \text{M5} \) of \( \text{M6} \):

\[ \text{M5} = 1. \quad \text{CcCqCrtCCCpqrsCsCq} \]
\[ \text{D1.1 = 2.} \quad \text{CCCpqrsCrCstCCursCCqCrtCqs} \]
\[ \text{D1.2 = 3.} \quad \text{CCCpqrsCqCrtCCursCCqCrtCqs} \]
\[ \text{D3.1 = 4.} \quad \text{CpCCCpqrsCqCrtCCursCCqCrtCqs} \]
\[ \text{D2.3 = 5.} \quad \text{CpCCCpqrsCqCrtCCursCCqCrtCqs} \]
\[ \text{D4.1 = 6.} \quad \text{CpCCCpqrsCqCrtCCursCCqCrtCqs} \]
\[ \text{D5.1 = 7.} \quad \text{CpCCCpqrsCqCrtCCursCCqCrtCqs} \]
\[ \text{D6.4 = 8.} \quad \text{CpCCCpqrsCqCrtCCursCCqCrtCqs} \]
\[ \text{D7.7 = 9.} \quad \text{CpCCCpqrsCqCrtCCursCCqCrtCqs} \]
\[ \text{D8.9 = \text{M6}.} \quad \text{CpCCCpqrsCqCrtCqs} \]

The following derivation from \( \text{M6} \) of Meredith’s known single axiom, \( \text{M1} \), thus completes the proof of the theorem:

\[ \text{M6} = 1. \quad \text{CsCCCpqrsCqCrtCqs} \]
\[ \text{D1.1 = 2.} \quad \text{CCCpqrsCqCrtCqs} \]
\[ \text{D2.1 = 3.} \quad \text{CpCCCpqrsCqCrtCqs} \]
\[ \text{D2.2 = 4.} \quad \text{CpCCCpqrsCqCrtCqs} \]
\[ \text{D3.1 = 5.} \quad \text{CpCCCpqrsCqCrtCqs} \]
\[ \text{D4.1 = 6.} \quad \text{CpCCCpqrsCqCrtCqs} \]
\[ \text{D6.4 = 8.} \quad \text{CpCCCpqrsCqCrtCqs} \]
\[ \text{D7.7 = 9.} \quad \text{CpCCCpqrsCqCrtCqs} \]
\[ \text{D8.9 = \text{M1}.} \quad \text{CpCCCpqrsCqCrtCqs} \]

In view of the foregoing, one might wonder if still more single axioms could be obtained by interchanging the penultimate letter \( q \) with \( X, Y, \) or \( Z \) in any of \( \text{M1-M6} \). The reader may confirm that the answer is negative by verifying that all eighteen formulas obtainable in this fashion are tautologies of the matrix:

\[
\begin{array}{c|ccc}
C & 1 & 2 & 3 \\
\hline
*1 & 1 & 3 & 3 \\
2 & 1 & 3 & 1 \\
3 & 1 & 1 & 1
\end{array}
\]

though the positive thesis \( Cpp \) obviously is not since \( C22 = 3 \).
References


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