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ON ULTRAFILTER LOGIC AND A MISSING AXIOM

Abstract
We examine a deductive system for ultrafilter logic, showing its soundness, completeness and independence. Ultrafilter logic is an extension of classical first-order logic by a generalised quantifier $\nabla$, whose intended interpretation is ‘almost all’. We modify a previous deductive system, adding a new axiom schema, related to alphabetic variants, which is shown to be sound and necessary to complete the calculus.

1. Introduction
We examine an axiomatisation for ultrafilter logic, showing its soundness, completeness and independence. Ultrafilter logic is an extension of classical first-order logic by a generalised quantifier $\nabla$, whose intended interpretation is ‘almost all’.

In this paper we examine a deductive system for ultrafilter logic, showing its soundness, completeness and independence. We modify a previous axiomatisation, adding a new axiom schema related to alphabetic variants, which is shown to be sound and to complete the calculus. The necessity of this new schema is established by showing the independence of the extended axiomatisation.

Ultrafilter logic intends to capture directly the intuition of a property holding for a large set of elements [12] and to serve as a precise basis for generic reasoning [4], [13]. For this purpose, one extends (conservatively) classical first-order logic by adding a new generalised quantifier $\nabla$ with intended interpretation ‘almost all’.
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The semantics of ultrafilter logic is given by means of ultrafilter structures, which expand first-order structures by ultrafilters on their universes. The semantical interpretation of $\nabla \forall \varphi$ is to be “$\varphi$ is almost universally valid”, i.e. the set of elements satisfying $\varphi$ is large in the sense of belonging to the given ultrafilter.

A deductive system for ultrafilter logic was proposed [4], [12]: it extends a deductive calculus for classical first-order logic by adding three axiom schemata, which code properties of ultrafilters. These schemata (and the corresponding ultrafilter properties) are as follows:

- $(\nabla \exists) \quad \nabla z \theta \rightarrow \exists z \theta \quad \{\text{large sets are nonempty}\}$,
- $(\nabla \land) \quad (\nabla z \psi \land \nabla z \theta) \rightarrow \nabla z (\psi \land \theta) \quad \{\text{intersections of large sets are large}\}$,
- $(\nabla \neg) \quad \neg \nabla z \theta \rightarrow \nabla z \neg \theta \quad \{\text{a set or its complement is large}\}$.

(The original formulation also had axiom $\forall z (\psi \rightarrow \theta) \rightarrow (\nabla z \psi \rightarrow \nabla z \theta)$, which can be derived from the others.)

For completeness, one needs a stronger version of schema $(\nabla \land)$, implicitly used in the previous proof attempt [4], [12]. This stronger version involves appropriate substitutions of variables: $(\nabla x \varphi (x) \land \nabla y \theta (y)) \rightarrow \nabla z (\varphi (z) \land \theta (z))$.

In this paper we extend the previous axiomatisation by a new axiom schema related to alphabetic variants, which is shown to be sound and to complete the calculus. We also establish the independence of the new axiomatisation, showing the necessity of its schemata.

To make the paper self contained we briefly review ultrafilter logic before examining the new axiom schema and showing its necessity for the proof of completeness.

2. Ultrafilter Logic

Ultrafilter logic extends classical first-order logic by a generalised quantifier $\nabla$, whose intended interpretation is ‘almost all’. In this section we briefly review ultrafilter logic: syntax, semantics and axiomatics.

We consider a fixed signature (logical type) $\lambda$ with a repertoire of symbols for predicates, functions and constants. We also consider a denumerably infinite set $V$ of new symbols for variables. We let $L(\lambda)$ be the
usual first-order language (with equality $\equiv$) of signature $\lambda$, closed under the propositional connectives, as well as under the quantifiers $\forall$ and $\exists$.

2.1. Syntax of $\nabla$

We use $L^\nabla(\lambda)$ for the extension of the usual first-order language $L(\lambda)$ obtained by adding the new operator $\nabla$.

The formulae of $L^\nabla(\lambda)$ are built by the usual formation rules and the following new variable-binding formation rule:

$(\nabla)$ for each variable $v \in V$, if $\theta$ is a formula in $L^\nabla(\lambda)$ then so is $\nabla v \theta$.

We shall also use the notations, for a formula $\varphi$ in $L^\nabla(\lambda)$:

- $\text{occ}(\varphi) (Fr(\varphi))$ for the set of variables with (free) occurrences in $\varphi$,
- $\varphi(v/t)$ for the result of substituting term $t$ for all the free occurrences of variable $v$ in $\varphi$.

Other syntactic notions, such as sentence, substitution, etc. can be appropriately.

2.2. Semantics of $\nabla$

The semantics of ultrafilter logic is provided by enriching first-order structures with ultrafilters and extending the usual definition of satisfaction to the generalised quantifier $\nabla$.

It is convenient to start with the more general case of complex structures.

A complex structure $A^F = (A, F)$ for signature $\lambda$ consists of a first-order structure $A$ for signature $\lambda$ together with a family $F \subseteq \mathcal{P}(A)$ of subsets of the universe $A$ of $A$.

We extend the usual definition of satisfaction of a formula $\varphi$ in a structure under an assignment $\sigma : V \to A$ to variables $(A^F \models \varphi[\sigma])$ as follows

$(\models \nabla)$ for a formula $\nabla v \theta$, we define $A^F \models \nabla v \theta[\sigma]$ iff the set

$\{ a \in A : A^F \models \theta[\sigma[v := a]] \}$

is in the given family $F$; where, as usual, $\sigma[v := a]$ is the assignment agreeing with $\sigma$ on every variable but $v$, and $\sigma[v := a](v) = a$. 
As usual, satisfaction of a formula depends only on the realisations assigned to its extra-logical symbols. Also, satisfaction of a formula hinges only on the values assigned to its free variables. So, we can employ the usual notation $A^F \models \varphi[a_1, \ldots, a_m]$, for $<a_1, \ldots, a_m> \in A^m$ and $fr(\varphi) = \{v_1, \ldots, v_m\}$.

We also have the familiar results concerning substitutions. We will need a special case: if no variable of term $t$ occurs bound in formula $\varphi$, then, for every variable $v:A^F \models \varphi(v/t)[\sigma]$ iff $A^F \models \varphi[\sigma[v:=A[t](\sigma)]]$, where $A[t](\sigma)$ is the value of $t$ in $A$ under assignment $\sigma$.

Other semantic notions, such as reduct, model $(A^F \models \tau)$, etc. are as usual.

We now relativise these ideas to the special class of ultrafilter structures.

An ultrafilter structure for signature $\lambda$ is a complex structure $A^U = (A, U)$ for signature $\lambda$, where family $U$ is a proper ultrafilter over the universe $A$ of $A$.

The corresponding notion of ultrafilter consequence is as expected: $\Gamma \models^U \tau$ iff $A^U \models \tau$ whenever $A^U \models \Gamma$, for every ultrafilter structure $A^U$.

2.3. Axiomatics of $\triangledown$

We will now set up a deductive system for our logic by adding schemata coding properties of ultrafilters to a calculus for classical first-order logic.

Consider first the following sets of formulae of $L^{\triangledown}(\lambda)$:

$$(\triangledown\exists) := \{\triangledown v\theta \rightarrow \exists v\theta : \theta \in L^{\triangledown}(\lambda)\};$$

$$(\triangledown\neg) := \{\neg\triangledown v\theta \rightarrow \triangledown v\neg\theta : \theta \in L^{\triangledown}(\lambda)\};$$

$$(\triangledown\land) := \{\triangledown (v\psi \land v\theta) \rightarrow \triangledown v(\psi \land \theta) : \psi, \theta \in L^{\triangledown}(\lambda)\}.$$ 

Consider also following set of formulae of $L^{\triangledown}(\lambda)$:

$$(\triangledown\alpha) := \{\triangledown v\theta \rightarrow \triangledown u\theta(v/u) : \theta \in L^{\triangledown}(\lambda), u \notin occ(\theta)\}.$$ 

Now, consider the set $A^{\triangledown}(\lambda)$ consisting of the generalisations of the formulae in the union $B^{\triangledown}(\lambda) := (\triangledown\exists) \cup (\triangledown\neg) \cup (\triangledown\land) \cup (\triangledown\alpha)$.

We set up a deductive system for our logic by adding the schemata in $A^{\triangledown}(\lambda)$ to a sound and complete deductive calculus $A(\lambda)$ for classical first-order logic, with Modus Ponens as the sole inference rule [6].
Thus, for a set $\Sigma$ of sentences and a formula $\varphi$ in $L^\nabla(\lambda)$, we have

$$\Sigma \vdash \nabla \varphi \iff \Sigma \cup A^\nabla(\lambda) \vdash \varphi.$$  

The following formulae are provable:

- $(\rightarrow \nabla) \forall z(\psi \rightarrow \theta) \rightarrow (\nabla z \psi \rightarrow \nabla z \theta)$  
  \{from $(\nabla \neg) \cup (\nabla \land) \cup (\nabla \exists)$\};
- $(\nabla^\alpha \land) (\nabla x \psi \land \nabla y \theta) \rightarrow \nabla z[\psi(x/z) \land \theta(y/z)]$, for $z \notin \operatorname{occ}(\psi \land \theta)$  
  \{from $(\nabla \alpha) \cup (\nabla \land)$\}.

Other usual deductive notions, such as (maximal) consistent sets, conservative extension, witnesses, etc. can be appropriately adapted.

For the proof of completeness, we need some properties of our deductive system, which can be established as in the case of classical first-order logic.

- Consider a consistent set $\Gamma$ of sentences in $L^\nabla(\lambda)$.
  1. (Henkin) There exists a consistent extension $\Delta$ of $\Gamma$ by (at most $|L^\nabla(\lambda)|$) new constants, where every existential sentence has a witness: if $\text{fr}(\exists v \theta) = \emptyset$, then $\Delta \vdash \nabla \exists v \theta \rightarrow \theta(v/c)$, for some constant $c$.
  2. (Lindenbaum) There exists a maximal consistent extension $\Sigma$ of $\Delta$ (over the same language).

3. The Logic of $\nabla$: Soundness, Completeness and Independence

We now establish some properties of our deductive system, namely soundness and completeness (with respect to ultrafilter structures) and independence of the axioms extending classical first-order logic.

3.1. Soundness

We first examine the soundness of our deductive system with respect to ultrafilter structure. As usual, soundness is easily established.

Indeed, the axioms in $A^\nabla(\lambda)$ code properties of ultrafilters, so they hold in every ultrafilter structure.

Clearly, the axioms in $(\nabla \exists) \cup (\nabla \land) \cup (\nabla \neg)$ code properties of ultrafilters, so they hold in every ultrafilter structure. As for $(\nabla \alpha)$, if a variable $u$ does not occur in $\theta$ ($u \notin \operatorname{occ}(\theta)$), we have $A^\mathcal{F} \models \theta[\sigma]$ iff $A^\mathcal{F} \models \theta(v/u)[\sigma]$. 
We thus have soundness of our deductive system with respect to ultrafilter consequence, since Modus Ponens preserves validity.

### 3.2. Completeness

We now examine the completeness of our deductive system with respect to ultrafilter structure. Completeness is usually harder, but we can adapt Henkin’s well-known proof for classical first-order logic, by providing an adequate ultrafilter by means of witnesses.

We proceed to outline how this can be done.

Given a consistent set $\Gamma$ in $L^\nabla(\lambda)$, extend it to a maximal consistent set $\Sigma$ in $L^\nabla(\lambda \cup C)$, where $C$ is a set of new constants providing witnesses for the existential sentences of $L^\nabla(\lambda \cup C)$.

Considering the set $T$ of variable-free terms of $L(\lambda \cup C)$, the canonical structure $H$ has universe $H := T/\approx$ where $t' \approx t''$ iff $\Sigma \vdash \nabla (t' \equiv t'')$.

Henkin’s inductive proof establishes for a sentence $\tau$ of $L(\lambda \cup C)$

$H \models \tau$ iff $\Sigma \vdash \tau$.

In our case, we need an extra inductive step to deal with the new quantifier $\nabla$. This can be handled as follows. Use the provable most sentences to form the family of large subsets of $H$:

$\Sigma^\nabla := \{ \Sigma(v|\theta) \subseteq H : \Sigma \vdash \nabla v \theta, fr(\nabla v \theta) = \emptyset \}$

where $\Sigma(v|\theta)$ is the represented within $\Sigma$ by formula $\theta$ of $L^\nabla(\lambda \cup C)$ with respect to a variable $v \in V$, in the sense

$\Sigma(v|\theta) := \{ t/\approx \in H : \Sigma \vdash \nabla v \theta(v/t) \}$.

Now, in view of our axioms, $\Sigma^\nabla \subseteq P(H)$ is closed under intersection (by $(\nabla \land)$ and $(\nabla \alpha)$) and $\emptyset \notin \Sigma^\nabla$ (by $(\nabla \exists)$). Thus, $\Sigma^\nabla$ has the finite intersection property and it can be extended to a proper ultrafilter $\mathcal{U} \subseteq P(H)$. We use this ultrafilter to expand the canonical structure $H$ to an ultrafilter structure $H^\mathcal{U} := (H, \mathcal{U})$ for $L^\nabla(\lambda \cup C)$.

We can now show, by induction, that for a sentence $\tau$ of $L^\nabla(\lambda \cup C)$

$H^\mathcal{U} \models \tau$ iff $\Sigma \vdash \nabla \tau$.

The inductive steps for propositional connectives as well as the quantifiers $\forall$ and $\exists$ are as in the classical proof.
Now, the inductive step for the new quantifier $\nabla$, namely

$$H^U \models \nabla v \theta \iff \Sigma \vdash \nabla v \theta,$$

for a sentence $\nabla v \theta$, follows from the crucial property

$$\Sigma(v(\theta)) \in \Sigma^U \iff \Sigma(v(\theta)) \in \mathcal{U}.$$

We thus have a Löwenheim-Skolem Theorem for our logical system:

- A consistent set of sentences of $L^\nabla(\lambda)$ has an ultrafilter model with cardinality at most that of its language.
- $\vdash \nabla$ is complete with respect to ultrafilter consequence: $\Gamma \models \nabla \tau$ if $\Gamma \models \nabla \tau$.

3.3. Independence

We shall now establish the independence of the axiom schemata of our deductive system extending classical first-order logic.

For each axiom schema $\chi$ in the set $B^\nabla(\lambda) = (\nabla \exists) \cup (\nabla \neg) \cup (\nabla \land) \cup (\nabla \alpha)$, we show several instances of $\chi$ that cannot be derived from the set $B^\nabla(\lambda) - \{\chi\}$.

For this purpose, we present for each axiom schema $\chi$ in the $B^\nabla(\lambda)$ a structure satisfying $B^\nabla(\lambda) - \{\chi\}$ but falsifying several instances of $\chi$.

For the cases $(\nabla \exists)$, $(\nabla \neg)$ and $(\nabla \land)$, we consider complex structures $A^F = (A, F)$, with universe $A$ and appropriate families $F \subseteq P(A)$, namely we take $F$ as $P(A)$, $\emptyset$ and $P(A) - \{\emptyset\}$ with $|A| > 1$, respectively.

$(\nabla \exists)$ For $[\nabla v \theta \rightarrow \exists v \theta] \in (\nabla \exists)$, take $F := P(A)$.

We see that $A^F$ satisfies $(\nabla \neg), (\nabla \land)$ and $(\nabla \alpha)$ trivially, but, for every instance $\neg v \equiv v$ of $\theta$, we have $A^F \not\models \nabla v \theta \rightarrow \exists v \theta[\sigma]$.

$(\nabla \neg)$ For $[\neg \nabla v \theta \rightarrow \nabla \neg \theta] \in (\nabla \neg)$, take $F := \emptyset$.

We see that $A^F$ satisfies $(\nabla \exists), (\nabla \land)$ and $(\nabla \alpha)$ vacuously, but, for every instance $v \equiv v$ of $\theta$, we have $A^F \not\models \neg \nabla v \theta \rightarrow \nabla \neg \theta[\sigma]$.

$(\nabla \land)$ For $[(\nabla v \psi \land \nabla v \theta) \rightarrow \nabla v (\psi \land \theta)] \in (\nabla \land)$, take $F := P(A) - \{\emptyset\}$ with $|A| > 1$. 
We see that $A^\mathcal{F}$ satisfies $(\forall \exists), (\forall \neg)$ and $(\forall \alpha)$, whereas, for distinct variables $u \neq v$ and instances $u \equiv v$ of $\psi$ and $\neg u \equiv v$ of $\theta$, we have $A^\mathcal{F} \models (\forall v \psi \land \forall v \theta)[\sigma]$ but $A^\mathcal{F} \nmid \forall v (\psi \land \theta)[\sigma]$.

We can see that the above complex structures also satisfy every formula in $((- \forall) := \{ \forall z (\psi \land \theta) \rightarrow (\forall z \psi \rightarrow \forall z \theta) : \psi, \theta \in L^\forall(\lambda) \})$.

For the case $(\forall \alpha)$, we consider functional structures of the form $A^\mathcal{F} = (A, f)$, where $A$ is a first-order structure, with universe $A$, and an appropriate function $f : V \rightarrow A$, and extend the usual definition of satisfaction (in a non-extensional manner) as follows:

$(\models \neg)$ for a formula $\forall v \theta$, we define $A^\mathcal{F} \models \forall v \theta[\sigma]$ iff the element $f(v)$ is in the set $\{ a \in A : A^\mathcal{F} \models \theta[\sigma[v := a]] \}$.

$(\forall \alpha)$ For $\forall v \theta \rightarrow \forall u \theta(v/u)$, with $u \notin \text{occ}(\theta)$, consider a first-order structure $A$ with $|A| > 1$, select two distinct elements $a' \neq a'' \in A$ and define $f : V \rightarrow A$ by $f(v_n) := a'$, for even $n$, and $f(v_n) := a''$, for odd $n$.

We easily see that $A^\mathcal{F}$ satisfies $(\forall \exists), (\forall \neg)$ and $(\forall \land)$ (so $(\forall \alpha)$), whereas, for each $i$ and odd $j$, a variable $u \notin \{ v_i, v_j \}$ and an assignment $\sigma : V \rightarrow A$ with $\sigma(v) = a'$, for instance $u \equiv v_i$, of $\theta$, we see that $A^\mathcal{F} \models \forall v \theta[\sigma]$ but $A^\mathcal{F} \nmid \forall v \theta(v_i/v_j)[\sigma]$.

Thus, for every set $A$ with $|A| > 1$ and each axiom schema $\chi$ in $B^\forall(\lambda)$, we have an expansion $A_\chi$ of a first-order structure $A$ with universe $A$ that satisfies $B^\forall(\lambda) \cup (\neg \chi) - \{ \chi \}$ but falsifies infinitely many instances of $\chi$.

Hence, each axiom schema $\chi$ in $B^\forall(\lambda) = (\forall \exists) \cup (\forall \neg) \cup (\forall \land) \cup (\forall \alpha)$ has infinitely many instances that cannot be derived from the set $B^\forall(\lambda) \cup (\neg \chi) - \{ \chi \}$.

We also mention that both complex and functional structures can be regarded as special cases of multicomplex structures.

A multicomplex structure $A^\mathcal{G}$ is an expansion of a first-order structure $A$ by a sequence $\mathcal{G} = (\mathcal{G}_n)$ of families $\mathcal{G}_n \subseteq \mathcal{P}(A)$ of subsets of the universe $A$. We extend the usual definition of satisfaction as follows:

$(\models \neg)$ for a formula $\forall v_n \theta$, we define $A^\mathcal{G} \models \forall v_n \theta[\sigma]$ iff the set $\{ a \in A : A^\mathcal{G} \models \theta[\sigma[v_n := a]] \}$ is in the family $\mathcal{G}_n$. 
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So, a complex structure $A^F = (A, F)$ corresponds to a multicomplex structure $A^G$ with constant sequence $G : G_n = F$, whereas a functional structure $A^f = (A, f)$ with $f : V \to A$ corresponds to a structure $A^P$ with families $G_n = \{X \subseteq A : f(v_n) \in X\}$.

4. Conclusion

We have examined a deductive system for ultrafilter logic, showing its soundness, completeness and independence. We have modified a previous axiomatisation, adding a new axiom schema, related to alphabetic variants, which is sound and completes the calculus. The necessity of this new schema has been established by showing the independence of the extended axiomatisation.

Ultrafilter logic intends to capture directly the intuition of a property holding for a large set of elements and to serve as a precise basis for generic reasoning. For this purpose, one extends (conservatively) classical first-order logic by adding a new generalised quantifier $\nabla$ with intended interpretation ‘almost all’ [4], [11], [13].

Our ultrafilter logic is a proper extension of classical first-order logic with compactness and Löwenheim-Skolem properties. The apparent conflict with Lindström’s results [9] is explained because we are using a non-standard notion of model (due to the ultrafilters in the models). This feature is likely to confer to ultrafilter logic some model-theoretic interest.

Ultrafilter logic is related to nonmonotonic reasoning [2], [10], which was one of its motivations [3], [11]. This logic also appears to have some interesting connections with inductive and empirical reasoning [7] as well as with fuzzy logic, suggesting the possibility of other applications for it [4], [13].

References


