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ON THE LATTICE OF ORTHOMODULAR LOGICS

Abstract

The upper part of the lattice of orthomodular logics is described.

In [1] and [2] Bruns and Kalmbach have described the lower part of
the lattice of varieties of orthomodular lattices. They proved that any va-
riety of orthomodular lattices essentially larger than variety generated by
\( a \) contains at least one of lattices \( b, c, d, e \). (By \( a, b, c, d, e, f, g \) we denote
the lattice depicted in respective picture.) Among those lattices only \( b \) is
modular. Let \( MO_n \) for \( n \in \omega \) and \( MO_\omega \) denote the modular ortholattice
with \( 2n \) (\( \omega \) respectively) pairwise incomparable elements and the bounds.
Then, lattices \( MO_2 \) and \( MO_3 \) are those depicted in \( a \) and \( b \) respectively.
Roddy [8] has proved that the varieties generated by ortholattices \( MO_n \)
and \( MO_\omega \) form an initial chain in lattice of varieties of modular ortholat-
tices. Therefore any variety of modular ortholattices which does not belong
to this chain is placed above it (i.e. contains \( MO_\omega \)). J. Malinowski [7] has
given a complete description of the lattice of subquasivarieties of \( MO_\omega \).

In this paper we describe all quasivarieties of orthomodular lattices
located below some of varieties generated by a single orthomodular lattice
from the set \( \{ b, c, d, e \} \).

Any logic generated by a class of orthomodular lattices is finitely
equivalent (J. Malinowski [6]). Moreover if \( C \) is a finitary finitely equiva-
 lent logic then the lattice of finitary strengthenings of \( C \) is dually iso-
 morphic with the lattice of subquasivarieties of the quasivariety of simple
matrices for \( C \) (J. Czelakowski [4]). Therefore we can apply theorem 4 to
describe the bottom of the lattice of logics (i.e. structural logical conse-
quence operations) generated by orthomodular lattices. For basic notions
and results in logics we refer to R. Wójcicki [9], for those in Universal
Algebra we refer to Burris Shankapanavar [3].
Let $K$ be a class of algebras, by $Q(K)$ (respectively $K$) we denote the least quasivariety (respectively variety) containing the class $K$. For every class $K$ of algebras an algebra $A \in K$ is subdirectly $K$ irreducible iff for every set $(A_i : i \in I)$ of algebras from $K$ if $A$ is a subdirect product of $(A_i : i \in I)$ then $A$ is isomorphic to $A_{i_0}$ for some $i_0 \in I$. If $K$ is a quasivariety then every algebra $A \in K$ is isomorphic to subdirect product of subdirectly $K$-irreducible algebras. A non trivial algebra $A$ will be called critical iff $A$ is subdirectly $Q(A)$-irreducible. A finite algebra is critical if and only if it does not belong to the quasivariety generated by its proper subalgebras.
Theorem 1. Every quasivariety of algebras is generated by its finitely generated critical algebras.

A class $K$ of algebras will be called locally finite if and only if every finitely generated algebra from $K$ is finite. If a class $K$ is generated by finite set of finite algebras then $K$ will be called finitely generated. By $\text{Con}(A)$ we shall denote the lattice of congruences of algebra $A$. If for every algebra $A \in K$ the lattice $\text{Con}(A)$ is distributive then the class $K$ will be called congruence distributive. An algebra $A$ with only two congruences (identity and full congruence) will be called simple. A variety $K$ will be called semi-simple if and only if every subdirectly irreducible algebra from the class $K$ is simple. Let $\theta_1, \theta_2 \in \text{Con}(A)$ for some algebra $A$, we will define the relation $\theta_1 \circ \theta_2$ on $A$ in the following manner: $a \theta_1 \circ \theta_2 b$ iff there is $c \in A$ such that $a \theta_1 c$ and $c \theta_2 b$. A class $K$ of algebras will be called congruence permutable if and only if for every $A \in K$ and $\theta_1, \theta_2 \in \text{Con}(A)$ the following equality holds $\theta_1 \circ \theta_2 = \theta_2 \circ \theta_1 = \theta_1 \lor \theta_2$.

Theorem 2. (W. Dziobiak, see J. Malinowski [7])

a) Let $K$ be a semi-simple, congruence permutable variety of algebras. Every finite critical algebra $A \in K$ is isomorphic to direct product of finitely many pairwise non-isomorphic simple algebras from $K$.

b) If $A = A_1 \times \ldots \times A_n$ is finite algebra, and there exist $i, j, k \in \{1, \ldots, n\}$ such that $A_i \hookrightarrow A_j$, $A_j \hookrightarrow A_k$, (the symbol $\hookrightarrow$ denotes “is isomorphic with a subalgebra of”) then $A$ is not critical.

c) If the algebras $A, B$ are non-isomorphic, finite, simple and do not contain one-element subalgebras then $A \times B$ is critical. □

An algebra $A = (A, \lor, \land, ')$ of the type $(2, 2, 1)$ will be called an ortholattice iff $(A, \lor, \land)$ is a bounded lattice, $'$ is anti-monotone complementation on $A$. An ortholattice which fulfills equality: $x \lor (x' \land (x \lor y)) = x \lor y$ will be called an orthomodular lattice. The variety of all orthomodular ortholattices will be denoted by $\text{OML}$. An ortholattice is modular (MOL for short) iff for $x \leq y$, $x \lor (y \land z) = y \land (x \lor z)$. The following theorem presents most important properties of orthomodular lattices. For proof of a) see for example Kalmbach [5] p. 79 and J. Malinowski [7], b) and c) come from Bruns, Kalmbach [2] and Roddy [8].
Theorem 3.

a) The variety \( OML \) is semi-simple and congruence permutable.

b) Every variety of orthomodular lattices properly containing \( MO2 \) contains \( b \) or \( c \) or \( d \) or \( e \).

c) Every variety of modular ortholattices different from any \( MOn \) for \( n \in \omega \) contains \( MO\omega \).

Theorem 4. Let \( 2 \) denote the two-element orthomodular lattice. The only critical algebras in the set \( Q \) – the set theoretic join of varieties generated by any single lattice from the set \( \{b, c, d, e\} \) are of the form:

\[
2, a, b, c, d, e, 2 \times a, 2 \times b, 2 \times c, 2 \times d, 2 \times e, a \times b, a \times c, a \times d, a \times e.
\]

Proof. It is easy to check that algebras from the set \( X = \{2, a, b, c, d, e\} \) are simple. There are no other simple algebras in \( Q \). Theorem 2 a) entails that any critical algebra in \( V \) is a product of one or two or more algebras belonging to \( X \). From theorem 2 c) any single element of \( X \) and any product of two elements of \( X \) is critical. Any choice of more than two elements from \( X \) contains at least two equal algebras, so any product of more than two algebras from \( X \) is not critical. \( \square \)

To the end we will apply the results achieved in theorem 4 to describe the upper part of the lattice of logics considered as a structural consequence operations. R. Wójcicki [9] contains an exhaustive monograph of the theory of logical consequences. For review of results in this area see also papers referred to below.

By orthomodular logic we will mean any logic determined by a class of logical matrices of the form \((A, 1)\), where \( A \) is an orthomodular lattice and 1 is unit element of \( A \). Any orthomodular logic is finitely equivalential (J. Malinowski [6]). J. Czelakowski [4] has proved that if \( C \) is a finitary finitely equivalential logic then the lattice of finitary strengthenings of \( C \) is dually isomorphic with the lattice of subquasivarieties of the quasivariety of simple matrices for \( C \). The facts above allow us to depict the lower levels of the lattice of orthomodular logics. Next levels seems too complicated to draw.
References


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