1. Introduction

Let \( Q-L \) be the least predicate extension of a normal extension \( L \) of \( S4 \) and \( BF \) be the Barcan formula \( \forall x \Box A(x) \supset \Box \forall x A(x) \). Ghilardi [3] showed that it is rare that \( Q-L \) is complete with respect to Kripke semantics.

On the other hand, if \( L \) is a subframe logic with the finite embedding property, we can show the completeness of \( Q-L + BF \) by the method of canonical models (cf. Lemma 3 [2], Theorem 3.9 [5]).

It is natural to ask whether \( Q-L + BF \) is complete if \( L \) is a subframe logic without finite embedding property.

Cresswell [4, chapter 14] described a proof due to Fine of the incompleteness of \( Q-S4M = Q-S4 + \Box \exists p \supset \exists \Box p + BF \) and asked whether \( Q-S4.3.1 + BF \) is complete or not.

In this note, we solve the problem negatively by proving the following theorem since \( Q-S4.3.1 + BF \) is in this interval.

**Theorem 1.** The modal predicate logics between \( Q-S4.1.4 \) and \( Q-S4.3Grz \) +BF are Kripke incomplete, where \( S4.1.4 = S4 + \Box(\Box (p \supset \Box p) \supset p) \supset (\Box \Box (p \supset p) \supset p) \supset p \) and \( S4.3Grz = S4.3 + \Box(\Box (p \supset \Box p) \supset p) \supset p \).

Throughout this note, we use the following terminology and notations. A preorder \( M = (M, R) \) is a reflexive and transitive binary relation. For
each \( u \in M \), the set \( \{ x \in M \mid xRu \} \) is denoted by \( u \downarrow \) and for each subset \( U \subseteq M \), \( U \downarrow = \bigcup_{u \in U} u \downarrow \). It is clear the relation defined by \( uRv \) and \( vRu \) is an equivalence relation. We call the equivalence class for it a cluster.

A modal propositional frame is a pre-order \( M \) and a modal predicate frame is the pair \( \langle M, U \rangle \) of a pre-order \( M = \langle M, R \rangle \) and a domain mapping \( U \). We denote \( \langle M, D \rangle \) instead of \( \langle M, U \rangle \) if \( U \) is a constant mapping \( U(u) = D(u \in M) \), and we call \( \langle M, D \rangle \) is of constant domain.

Let \( \langle M, U \rangle \) be a modal predicate frame and \( L \) a language of modal predicate logic. \( L[U(u)] \) denotes the language obtained from \( L \) by adding all the constant symbols \( c \) for \( c \in U(u) \). A valuation \( \models \) on \( \langle M, U \rangle \) is a binary relation between each \( a \in M \) and atomic \( L[U(a)] \) sentences. We can extend \( \models \) to the relation between each \( a \in M \) and \( L[U(a)] \) sentences in the usual way. We also denote the extended relation by \( \models \). A modal predicate model is the triple \( \langle M, U, \models \rangle \), where \( \langle M, U \rangle \) is a frame and \( \models \) is a valuation on \( \langle M, U \rangle \).

A modal predicate sentence \( A \) is said to be valid in a modal predicate frame \( \langle M, U \rangle \) if \( a \models A \) holds for every \( a \in M \) and every valuation \( \models \) on \( \langle M, U \rangle \), and a modal predicate logic \( L \) is said to be

Our proof of Theorem 1 is a modification of that for \( Q\text{-}S4M + BF \) in [4]. That is, we show that the formula

\[
A = \Box(\Box\forall x(p(x) \lor \Box p(x)) \lor \forall xp(x)) \lor (\Box\Box\Box\forall xp(x) \lor \forall xp(x)),
\]

has the following properties.

(a) \( A \) is valid in every modal predicate frame validating \( Q\text{-}S4.1.4 \).
(b) \( A \) is not a theorem of \( Q\text{-}S4.3Grz + BF \).

Note that \( A \) is obtained from the axiom of \( S4.1.4 \) by inserting some quantifiers \( \forall x \). This fact enables us to construct a valuation refuting the axiom of \( S4.1.4 \) from that for \( A \), and (a) follows from this.

On the other hand, \( S4.1.4 \) is the subframe logic axiomatized by the subframe formula for the propositional frame \( M_0 = \langle \{0,1,2\}, R_0 \rangle \) in Figure 1.
In the proof of (b), we use a Kripke model whose frame part is obtained from the frame $M_0$ by substituting the proper cluster by an infinite chain. This suggests that we can prove the incompleteness of other subframe logics without the finite embedding property by a similar method.

2. Proof of (a)

Firstly, we prove S4.1.4 is a subframe logic (cf. Fine [2]). A partial function $f$ from $M$ to $N$ is said to be a subreduction if it is a p-morphism from its domain to $N$.

**Lemma 2.** Let $\langle M, U \rangle$ be a modal predicate frame. Then, the following are equivalent.

1. S4.1.4 is valid in $\langle M, U \rangle$.
2. There is no subreduction from $M$ to the frame $M_0$ in Figure 1.

**Proof.** $\Rightarrow$ Suppose there is a subreduction $f$ from $M$ to $M_0$. We define a valuation on $\langle M, U \rangle$ as follows:

$$u \models p \iff u \in f^{-1}(1) \text{ or } u \not\in f^{-1}(0) \downarrow.$$ 

Pick $u_0 \in f^{-1}(0)$. We show $u_0 \not\models \Box(\Box(p \supset \Box p) \supset p) \supset (\Box \Box \Box p \supset p)$. First note that $u \models \Box p$ if and only if $u \not\in f^{-1}(0) \downarrow$ since the complement of $f^{-1}(0) \downarrow$ is upward closed.

So, $u \models \Box(\Box(p \supset \Box p) \supset p)$ if and only if $u \not\in f^{-1}(0) \downarrow$ since $v \not\models p \supset \Box p$ for $v \in f^{-1}(1)$. Hence, $u \models \Box(\Box(p \supset \Box p) \supset p)$ for every $u$ and $u_0 \models \Box(\Box(p \supset \Box p) \supset p)$.
\( \Box p \supset p \).

On the other hand, since every \( u \in f^{-1}(0) \) is below some \( v \in f^{-1}(2) \), \( u \models \Diamond \Box p \) holds. Since \( u \models \Box p \) holds for every \( u \notin f^{-1}(0) \), \( u_0 \models \Box \Box \Box p \).

Therefore, \( u_0 \not\models \Box (\Diamond (p \supset \Box p)) \supset (\Box \Diamond \Diamond p \supset p) \).

\( \Leftarrow \) Suppose \( u_0 \models \Box (\Diamond (p \supset \Box p)) \supset (\Box \Diamond \Diamond p \supset p) \) and \( u_0 \not\models p \). We define a partial function \( f \) as follows:

\[
 f(u) = \begin{cases} 
 0 & \text{if } u_0Ru \text{ and } u \not\models p \\
 1 & \text{if } u_0Ru \text{ and } u \not\models p \supset \Box p \\
 2 & \text{if } u_0Ru \text{ and } u \models \Box p \\
 \text{undefined} & \text{otherwise}
\end{cases}
\]

We show \( f \) is a subreduction.

To prove that \( uRv \) implies \( f(u)Rf(v) \) for \( u, v \in \text{dom}(f) \), we may assume \( f(u) = 2 \) and \( f(v) \neq 2 \). \( uRv \) cannot holds since \( u \models \Box \Box \Box p \) and \( v \not\models \Box p \). Hence, \( f \) is order preserving.

Therefore \( f \) is a subreduction. \( \quad \text{q.e.d.} \)

If \( A \) is not valid in \( \langle M, U \rangle \), then there is a valuation \( \models \) on \( \langle M, U \rangle \), \( u_0 \in M \) and \( a_0 \in U(u_0) \) such that the following hold:

\[
\begin{align*}
 u_0 &\models \Box (\Diamond \forall x(p(x) \supset \Box p(x)) \supset \forall xp(x)), \\
 u_0 &\models \Box \Diamond \forall xp(x), \\
 u_0 &\not\models p(a_0).
\end{align*}
\]

We claim we can inductively define \( u_n, v_n \in M \) (\( n \geq 1 \)) and \( a_n \in U(v_n) \) (\( n \geq 1 \)) satisfying the following:

\[
\begin{align*}
 u_{n-1} &\ R \ v_n \ R \ u_n, \\
 v_n &\models p(a_n), \\
 v_n &\not\models \Box p(a_n), \\
 u_n &\not\models p(a_n).
\end{align*}
\]

Suppose \( u_{n-1} \) and \( a_{n-1} \) are defined. Since \( u_{n-1} \not\models p(a_{n-1}) \) implies \( u_{n-1} \not\models \forall xp(x) \), we can pick \( v_n \) and \( a_n \in U(v_n) \) such that \( u_{n-1} \ R \ v_n \) and
\(v_n \not\models p(a_n) \supset \Box p(a_n)\) by \(u_0 \mathcal{R} u_{n-1}\) and (1). By (6), there exists \(u_n\) such that \(v_n \mathcal{R} u_n\) and \(u_n \not\models p(a_n)\).

By (2), there exists \(w_n\) such that \(v_n \mathcal{R} w_n\) and \(w_n \models \Box \forall x p(x)\) for every \(n\).

Note that \(u_n \neq v_n\) since \(v_n \models p(a_n)\) and \(u_n \not\models p(a_n)\). Similarly, \(w_m \mathcal{R} v_n\) for every \(m, n\) since \(v_n \not\models \Box p(a_n)\) and \(w_m \models \Box \forall x p(x)\).

Case 1) There exist \(v_m = v_n\) for some \(m < n\).

In this case, \(v_m \mathcal{R} u_m \mathcal{R} v_n = v_m\) holds. So, we can define a subreduction \(f\) as follows:

\[
\text{dom}(f) = \{u_m, v_m, w_m\}, f(v_m) = 0, f(u_m) = 1, f(w_m) = 2.
\]

Case 2) The elements of \(\{v_m\}\) are all distinct.

In this case, we can define a subreduction \(f\) as follows:

\[
\text{dom}(f) = \{v_n, w_n \mid n \geq 1\}, f(v_{2n-1}) = 0, f(v_{2n}) = 1, f(w_n) = 2.
\]

So, in both cases \(\langle \mathcal{M}, U \rangle\) does not validate \(S4.1.4\) by Lemma 2.

This completes the proof of (a).

3. Proof of (b)

In this section, \(\langle \mathcal{M}, D, \models \rangle\) denotes the Kripke model of constant domain defined as follows:

- \(\mathcal{M} = \langle \omega + 1, \leq \rangle\),
- \(D = \omega\),
- \(m \models p(m) \leftrightarrow n = m + 1\) for all \(m < \omega\) and \(n \in D\),
- \(\omega \models p(\omega)\) for all \(n \in D\).

Note that, if we delete the topmost element \(\omega\) from our models, we obtain a model essentially equal to that for \(Q\text{-SM} + BF\). We slightly change the valuation because the proof in [4] contains a minor error.\(^4\)

\(^4\)In fact, Lemma 14.12 in [4] does not hold for \(m = 0, n > 0\). A counter example is \(a = \exists x \Box \neg \psi(x)\).
Lemma 3. [cf. Lemma 14.12 [4]] Let $B(x_1, x_2, \ldots, x_k)$ be a formula with free variables $x_1, x_2, \ldots, x_k$.

Then, for every $m \leq n < \omega$ and $k$-tuples $(p_1, p_2, \ldots, p_k), (q_1, q_2, \ldots, q_k) \in D^k$ satisfying the condition

\[ p_i \leq m, q_i \leq n \]

or

\[ p_i - q_i = m - n, \]

then

\[ m \models B(p_1, p_2, \ldots, p_k) \text{ if and only if } n \models B(q_1, q_2, \ldots, q_k). \]

Proof. By induction on the complexity of $B(x_1, x_2, \ldots, x_k)$. We only treat the case $B(x_1, x_2, \ldots, x_k)$ is of the form $\forall x B'(x, x_1, x_2, \ldots, x_k)$.

$\Rightarrow$) We prove the contraposition. Suppose

\[ n \not\models \forall x B'(x, q_1, q_2, \ldots, q_n). \]

Then there exists $q \in D$ such that

\[ n \not\models B'(q, q_1, q_2, \ldots, q_n). \]

Put

\[ p = \begin{cases} 
  m & (q \leq n) \\
  m + (q - n) & (q > n),
\end{cases} \]

Then, by induction hypothesis,

\[ m \not\models B'(p, p_1, p_2, \ldots, p_n), \]

and

\[ m \not\models \forall x B'(x, p_1, p_2, \ldots, p_n). \]

$\Leftarrow$) Change the role of $m, p, p_1, \ldots, p_k$ and $n, q, q_1, \ldots, q_k$ in the above case.

q.e.d.
Lemma 4. Every instance of $\text{S4.3Grz}$ is true in $\langle M, D, \models \rangle$.

Proof. Since $\langle M, D \rangle$ is linear and of constant domain, $\Box(\Box p \supset q) \lor \Box(\Box q \supset p)$ and $BF$ are valid in $\langle M, D \rangle$. So, it is enough to show the case $\text{Grz} = \Box(\Box p \supset \Box q) \supset p$.

Let $C = \Box(\Box B \supset \Box B) \supset B$ be an $L[D]$ instance of $\text{Grz}$ and $p_1, p_2, \ldots, p_k$ be the constant symbols occurring in $B$.

Put $N = \max\{p_1, p_2, \ldots, p_k\}$. Then, by Lemma 3, for every $N \leq m, n < \omega$ the following holds:

$m \models B \iff n \models B$.

Suppose $C$ is not true at some $m < \omega$. Then there exists sequences $\{m_i\}$ and $\{n_i\}$ such that the following hold.

$m_i < n_i < m_i + 1 \ (i \in \omega)$,

$m_i \not\models B, n_i \models B, n_i \not\models 2B$.

This contradicts to Lemma 3. q.e.d.

To prove the assertion of (b), we have only to show $0 \not\models A$ by Lemma 4.

Since $n \models p(n + 1)$ and $n + 1 \not\models p(n + 1)$ for every $n < \omega$, $n \not\models \forall x(p(x) \supset \Box p(x))$. So, $n \models \Box \forall x(p(x) \supset \Box p(x)) \supset \forall x(p(x))$ holds. On the other hand, $\omega \models \forall x(p(x))$ implies $\omega = \Box \forall x(p(x) \supset \Box p(x)) \supset \forall x(p(x))$. Hence, $0 \models \Box(\Box \forall x(p(x) \supset \Box p(x)) \supset \forall x(p(x)))$ holds.

Moreover, since $\omega$ is the maximum in $M$, $0 \models \Box \Diamond \Box \forall x p(x)$ holds. Therefore $0 \not\models A$ by $0 \not\models \forall x p(x)$.

This completes the proof of (b).

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