AXIOMATIZATION OF THE LOGICS DETERMINED BY FINITE RELATIONAL SYSTEMS OF NATURAL NUMBERS WITH IDENTITY

In this paper we show that the logics determined by relational systems $N_n = < N_n; = >$, where $N_n = \{1, 2, \ldots, n\}$ are axiomatizable.

1. Let $N_n = < N_n; = >$ (for any $n \in N = \{1, 2, \ldots\}$) be the relational system with identity such that: $N_n = \{1, 2, \ldots, n\}$. By $L_{N_n}$ we shall denote the first - order predicate language for $N_n$. The set of all formulas $F_{N_n}$ of $L_{N_n}$ is the smallest set satisfying the conditions:

a. $(x_i = x_j) \in F_{N_n}$ (for $i, j \in N$)

and

b. If $A, B \in F_{N_n}$, then $\neg A, A \circ B, Qx_k A \in F_{N_n}$, where $\circ \in \{\land, \lor, \rightarrow, \equiv\}$ and $Q \in \{\forall, \exists\}$.

The symbol $E(N_n)$ will denote the set of all formulas of language $L_{N_n}$, which are true in $N_n$.

We shall introduce now some particular notions.

Formulas $A$ and $B$ are similar (in symbols $A \approx B$) if they differ at most on bound variables.

By $x^n / x_k$ we denote the substitution for individual variables. By $[\ldots]_n$ we mean an operation which increases the indices of bound variables:

$$[\land A]_i = \land_{x_k + t} [A(x_k / x_{k + t})]_n,$$

where $t$ is the greatest index among all the indices of all individual variables occurring in $A$. For the logical connectives and for atomic formulas this operation is invariant.

2. The symbols $P_k^m$ ($k, m \in N$) are $m$ - ary predicate letters. By an atomic formula we mean $P_k^m(x_{i_1}, \ldots, x_{i_m})$. The set of all atomic formulas is denoted by $At$. The set of all logical schemes $SL$ is the smallest set such that:
a. $At \subseteq SL$

and

b. If $\alpha, \beta \in SL$, then: $\neg \alpha, \alpha \circ \beta, Q_{x_k} \alpha \in SL$.

Let $E^*$ be the set of functions of substitution for atomic schemes (for details see [3]). Any function $s : At \rightarrow SL$ can be extended to an endomorphism $h^* : SL \rightarrow SL$.

For every relational system $N_n$ we define now the set $V_{N_n}$ of functions as follows:

$v \in V_{N_n}$ if and only if

a. $v : At \rightarrow F_{N_n}$,

b. $Vf(v(\alpha)) \equiv Vf(\alpha)$,

c. $v(\alpha(x_k/x_m)) \approx [v(\alpha)]_{k+m}(x_k/x_m)$

for every $\alpha \in At \subseteq SL$, where $Vf(\alpha)$ denotes the set of all free variables occurring in $\alpha$.

Note that every function $v \in V_{N_n}$ can be extended to a homomorphism $h^v : SL \rightarrow F_{N_n}$.

A scheme $\alpha \in SL$ is a predicate tautology of the relational system $N_n = < N_n, =>$ if and only if for every $v \in V_{N_n}$ : the formula $h^v(\alpha)$ is true in $N_n$.

The set of all predicate tautology of $N_n$ is denoted by $L(N_n)$ and called the predicate logic of the system $N_n$.

The symbol $r_\circ$ denotes the modus ponens rule and the symbol $r_\land$ the rule of generalization. The rule $r_\ast$ of substitution for atomic formula is defined as follows:

$< \{ \alpha \}; \beta > \in r_\ast$ if and only if there exists $s \in E^*$ such that $\beta = h^s(\alpha)$.

By $L_2$ we denote the set of all logical schemes which are theorems of the classical first - order predicate logic.

Let now $R = \{ r_\circ, r_\land, r_\ast \}$, then the set $Cn_R(X)$ is the smallest set containing $X \subseteq SL$ and closed under each of the rule $r \in R$.

3. Let now $E_m$ be the set of all equivalence relations over the set $N_m = \{1, 2, \ldots, m\}$. For every $e \in E_m$ we define the function $f_m^e$ as follows:

$f_m^e(< i, j >) = \left\{ \begin{array}{ll}
P^2_1(x_i, x_j) & \text{if } < i, j > \in e \\
\neg P^2_1(x_i, x_j) & \text{if } < i, j > \notin e.\end{array} \right.$

We shall use the symbols: $\prod_{t \in \{a_1, \ldots, a_k\}} \alpha_t$ and $\sum_{t \in \{a_1, \ldots, a_k\}} \alpha_t$ for $\alpha_{a_1} \land \ldots \land \alpha_{a_k}$ and $\alpha_{a_1} \lor \ldots \lor \alpha_{a_k}$ respectively.
For every \( n \in \mathbb{N} \) such that \( n > 1 \) we consider the following logical schemes (where \( P(x, y) \) denote the abbreviation of the scheme \( P^2_k(x, x_j) \)):

\[
\begin{align*}
\text{(S1)} \quad & \land_{x \neq y} [P(x, y) \rightarrow P(y, x)] \\
\text{(S2)} \quad & \land_{x} P(x, x) \rightarrow \land_{x \neq y} \land[P(x, y) \land P(y, z) \rightarrow P(x, z)] \\
\text{(Sh}_1^{\text{m}}) \quad & \land_{x} P(x, x) \land \bigvee_{x \neq y} \neg P(x, y) \rightarrow \land_{x_1 \neq \ldots \neq x_{n-1}} \bigvee_{x_{n-1}} \neg P(x_1, y) \land \ldots \land \neg P(x_{n-1}, y) \\
\text{(S4)} \quad & \land_{x} \ldots \land \left[ \land_{x_1 \neq \ldots \neq x_{n+2}} [P(x_1, x_{n+2}) \equiv P(x_2, x_{n+2})] \lor \ldots \lor [P(x_1, x_{n+2}) \equiv P(x_{n+1}, x_{n+2})] \lor \land_{x_1 \neq \ldots \neq x_{n+2}} [P(x_2, x_{n+2}) \equiv P(x_3, x_{n+2})] \lor \ldots \lor [P(x_2, x_{n+2}) \equiv P(x_{n+1}, x_{n+2})] \lor \ldots \lor [P(x_{n+1}, x_{n+2}) \equiv P(x_{n-1}, x_{n+2})] \lor [P(x_{n+1}, x_{n+2}) \equiv P(x_{n+1}, x_{n+2})]] \\
\text{(Sm}_5^{\text{m}}) \quad & \land_{x} P(x, x) \land \bigvee_{x \neq y} \neg P(x, y) \rightarrow \land_{x} \ldots \land [P(x, y) \rightarrow \ldots \land (P^m_k(x_1, \ldots, x_m)(x_i / x) \equiv P^m_k(x_1, \ldots, x_m)(x_i / y))] \quad k, m = 1, 2, \ldots, i = 1, 2, \ldots, m. \\
\text{(Sm}_6^{\text{m}}) \quad & \land_{x} P(x, x) \land \bigvee_{x \neq y} \neg P(x, y) \rightarrow (D^m_k \lor C^m_k),
\end{align*}
\]

where \( D^m_k = \land_{x_1 \neq \ldots \neq x_m} [P^m_k(x_1, \ldots, x_m) \equiv (\neg P(x_1, x_1) \lor \ldots \lor \neg P(x_m, x_m))] \) and

\[
C^m_k = \sum_{\emptyset \neq \mathcal{X} \subseteq \mathcal{E}_m} \prod_{x_1 \neq \ldots \neq x_m} f^m_i(<i, j>),
\]

\( k, m = 1, 2, \ldots \)

Let \( Ax_{N_n} \) be the set of all logical schemes: \( (S1), (S2), (S_{3}^{n-1}), (S4), (S_{5}^{m}), (S_{6}^{m}) \), \( k, m = 1, 2, \ldots \).
Then we have a theorem, whose proof is an adaptation of the method presented in [1].

**Theorem.** $L(N_n) = Cn_R(L_2 \cup Ax_{N_n})$.

**References**


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