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EXPLICIT AND IMPLICIT DEFINABILITY IN MODAL AND RELATED LOGICS

We consider various versions of the Beth definability property for propositional normal modal logics, and also for superintuitionistic and relevant logics. We discuss interrelations of these properties, and find their algebraic equivalents in case of modal and superintuitionistic logics.

Let $x$, $q$, $q'$ be disjoint lists of variables not containing $y$ and $z$, $A(x,q,y)$ a formula.

We say that a logic $L$ has the projective Beth property $PB1$. If $\vdash L A(x,q,y) & A(x,q',z) \rightarrow (y \leftrightarrow z)$ implies $\vdash L A(x,q,y) \rightarrow (y \leftrightarrow B(x))$ for some formula $B(x)$.

Further, $L$ has the projective Beth property $PB2$. If $A(x,q,y), A(x,q',z) \vdash L y \leftrightarrow z$ implies $A(x,q,y) \vdash L y \leftrightarrow B(x)$ for some $B(x)$.

$L$ has the Beth Property $B1$. If $\vdash L A(x,y) & A(x,z) \rightarrow (y \leftrightarrow z)$ implies $\vdash L (A(x,y) \rightarrow (y \leftrightarrow B(x)))$ for a suitable $B(x)$.

$L$ has the Beth Property $B2$. If $A(x,y), A(x,z) \vdash L y \leftrightarrow z$ implies $A(x,y) \vdash L y \leftrightarrow B(x)$ for some $B(x)$.

CIP. If $\vdash L A(x,q) \rightarrow B(x,r)$ then there is a formula $C(x)$ such that both $\vdash L A(x,q) \rightarrow C(x)$ and $\vdash L C(x) \rightarrow B(x,r)$.

IPD. If $A(x,q) \vdash L B(x,r)$ then there is a $C(x)$ such that $A(x,q) \vdash L C(x)$ and $C(x) \vdash L B(x,r)$.

In [6] a diagram of interrelations between the properties B1, B2 and interpolation properties CIP and IPD was found for normal modal logics (n.m.l.). It was proved that B1 is equivalent to CIP, and implies B2 and IPD; the properties IPD and B2 are independent.
Theorem 1. In the family of normal modal logics

(i) $PB_1$ is equivalent to $CIP$ and to $B_1$,
(ii) $PB_1$ implies the conjunction of $B_2$ and $IPD$, but the converse does not hold,
(iii) the conjunction of $B_2$ and $IPD$ implies $PB_2$,
(iv) $PB_2$ implies $B_2$ but the converse does not hold.

Since all normal extensions of $K_4$ have $B_2$ [6], we get

Theorem 2. For every n.m.l. containing $K_4$, $IPD$ implies $PB_2$.

It is an open problem, if $PB_2$ implies $IPD$ in modal logics. The answer is positive for normal extensions of $S_5$.

Theorem 3. For each n.m.l. containing $S_5$, $PB_2$ implies $CIP$, therefore, all properties $PB_1$, $B_1$, $CIP$, $IPD$, $PB_2$ are equivalent in $NE(S_5)$.

It was proved in [5] that just four logics in $NE(S_5)$ satisfy $IPD$ and $CIP$. At the same time, all extensions of $S_5$ have $B_2$.

$ES^*$. For any $A, B$ in $V(L)$, for any monomorphism $\alpha : A \to B$ and for any $x \in B - \alpha(A)$, such that $\{x\} \cup \alpha(A)$ generates $B$, there exist $C \in V$ and monomorphisms $\beta : B \to C$ and $\gamma : B \to C$ such that $\beta \alpha = \gamma \alpha$ and $\beta(x) \neq \gamma(x)$.

Now we define a stronger property

$SES$. For any $A, B$ in $V(L)$, for any monomorphism $\alpha : A \to B$ and for any $x \in B - \alpha(A)$ there exist $C \in V$ and monomorphisms $\beta : B \to C$ and $\gamma : B \to C$ such that $\beta \alpha = \gamma \alpha$ and $\beta(x) \neq \gamma(x)$.

Theorem 4. N.m.l. $L$ satisfies $PB_2$ iff $V(L)$ has $SES$.

Due to the deduction theorem, $B_1$ is equivalent to $B_2$, and $CIP$ is equivalent to $IPD$ for any superintuitionistic logic (s.i.l.). Every s.i.l. has $B_1$ [2] but only eight s.i.l. have $CIP$ [4]. It is easy to see that $CIP$ implies $PB_1$, and $PB_1$ implies $B_1$. Further, $CIP$ in s.i.l. $L$ is equivalent to the amalgamation property $AP$ in $V(L)$. We can prove that $B_1$ is equivalent to $ES^*$, and $PB_1$ is equivalent to $SES$. It is an open problem: How many s.i.l. have $PB_1$?
It was proved in [9] that the basic relevant logics, among them \(\text{E}\) and \(\text{R}\), have neither CIP nor Beth definability properties. Nevertheless, CIP holds in \(\text{OR}\) which is \(\text{R}\) without distributivity axiom [7].

Only some weak forms of the deduction theorem hold in relevant logics [3], [8]. On this reason, the usual implication from CIP to IPD holds for extensions of \(\text{E}\) if the language includes a propositional constant \(t\) (“the strongest truth”), where \(\vdash_L\) denotes the deducibility with modus ponens and adjunction. To preserve a standard proof of the Beth property from CIP, we need an intensional conjunction \(\circ\) (that is commutative and associative) and the following definition

\[
PBI'. \quad \text{If } \vdash_L A(x, q, y) \circ A(x, q', z) \rightarrow (y \leftrightarrow z) \text{ then } \vdash_L A(x, q, y) \rightarrow (y \leftrightarrow B(x)) \text{ for some } B(x).
\]

Thus CIP implies PBI’ in extensions of \(\text{R}\). The equivalence of IPD to \(\text{AP}\) holds for all extensions of \(\text{E}\) [1], and the equivalence of CIP in \(L\) to \(\text{SAP}\) in \(V(L)\) holds for extensions of \(\text{R}\) (or of the fragment of \(\text{R}\) with \(t\), \&, \(\rightarrow\) and \(\circ\)).

References


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