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# A MAXIMAL LATTICE OF IMPLICATIONAL LOGICS[1]

The problem about the finding of the unified foundation for the classification of implicational logics was raised by V.A.Smirnov in his article [10]. He suggested to classify implicational logics (1) in relation of the form of deduction theorem and (2) based only on structural rules. V.A.Smirnov pays attention to the very important problem, namely, that both method of classification do not include classical logic. In the first case the deduction theorem which takes place for an implicational fragment $\mathbf{H}_{\to}$ of intuitionistic propositional logic **H** is true also for an implicational fragment $\mathbf{TV}_{\to}$ of classical logic **TV**. And then in this case no distinction between $\mathbf{H}_{\to}$ and $\mathbf{TV}_{\to}$ logics is made. In the second case it does not exist any structural rule which would provide the transition from $\mathbf{H}_{\to}$ to $\mathbf{TV}_{\to}$. This transition is usually realized due to the admission of Pearce's law

$$\mathbf{P}.\ ((\mathrm{p}\to \mathrm{q}) \to \mathrm{p}) \to \mathrm{p}.$$

But there exists no structural rule corresponding to this formula. See also the problem with formula **P** in [3].

Our proposal of resolving the problem consists in yielding such logical construction which comprises implicational logics in question. Moreover, application the simplest operations to the construction would allow to generate new logics and even infinite classes of logics. Let us take as primitive objects from which we will build construction the following implicative formulas:

**I**. $p \to p$
**B**. $(q \to r) \to ((p \to q) \to (p \to r))$
**C**. $(p \to (q \to r)) \to (q \to (p \to r))$
**W**. $(p \to (p \to q)) \to (p \to q)$
$\mathbf{K_1}$. $(p \to q) \to (r \to (p \to q))$.

The operations are nothing else but two following inference rules:

[1] A full version will be published in Studia Logica.

R1. *Modus ponens*: from $A \to B$ and $A$ it follows $B$.
R2. *Substitution* of propositional variables.

For example, formula $\mathbf{K_1}$ is the result of simultaneous substitution in formula $\mathbf{K}$:

$$\mathbf{K}.\ p \to (q \to p),$$

where $p \to q$ is substituted instead of $p$, and $r$ is substituted instead of $q$, i. e. $p/p \to q$ and $q/r$.

Now it is the main point. The fundamental demand to our primitive objects, i.e. to the implicative formulas, is the property of their independence from each other.

Theorem 1. *The set of formulas* **I**, **B**, **C**, **W**, $\mathbf{K_1}$ *is an independent axiomatization of* $\mathbf{H}_{\to}$.

The problem which now arises is similar to Smirnov's problem. In our case the problem is formulated in the following way: is there formula **X** such that **I**, **B**, **C**, **W**, $\mathbf{K_1}$, **X** is an independent axiomatization of $\mathbf{TV}_{\to}$ [4]?

Note that the Pirce's law is not suitable formula instead of **X** since **I**, **B**, **C** + **P** is already axiomatization of $\mathbf{TV}_{\to}$, i. e. **I**, **B**, **C**, **P** $\vdash$ **W**, **K**, and it means that the set of formulas **I**, **B**, **C**, **W**, **K**, **P** is not independent axiomatization of $\mathbf{TV}_{\to}$. Thus we must weaken the formula **P**. In November 1992 I found a suitable weakening of formula **P**:

$$\mathbf{X_1}.\ ((p \to q) \to ((r \to r) \to (p \to q))) \to (\mathbf{W_1} \to \mathbf{P_1}),$$

where $((p \to q) \to ((r \to r) \to (p \to q)))$ is an substitutional instance of $\mathbf{K_1}$: $r/r \to r$; $\mathbf{W_1}$ is an substitutional instance of $\mathbf{W}$: $p/p \to q, q/r$; $\mathbf{P_1}$ is an substitutional instance of $\mathbf{P}$: $p/p \to q, q/r$.

Theorem 2. *The set of formulas* **I, B, C, W,** $\mathbf{K_1}$, $\mathbf{X_1}$ *is an independent axiomatization of* $\mathbf{TV}_{\to}$.

Thus the problem of suitable extending of $\mathbf{H}_{\to}$ to $\mathbf{TV}_{\to}$ is solved. But we can simplify the formula $\mathbf{X_1}$. Note that formula $\mathbf{W_1} \to \mathbf{P_1}$ is an substitutional instance of formula **D**:

$$\mathbf{D}.\ ((p \to q) \to q) \to ((q \to p) \to p).$$

Now we can represent the formula $\mathbf{X_2}$:

$$(p \to ((q \to q) \to p)) \to \mathbf{D},$$

where $(p \to ((q \to q) \to p))$ is a substitutional instance of the formula **K**: $q/q \to q$.

Theorem 3. *The set of formulas* **I, B, C, W, K$_1$**, **X$_2$** *is an independent axiomatization of* $\mathbf{TV}_\to$ *(compare with [5]).*

At last, Slaney and Bunder in [9] proposed another formula (due to Meyer and Parks [8]) instead of **X**:

$$\mathbf{X_3}.\ ((((p \to q) \to q) \to p) \to r) \to (((((q \to p) \to p) \to q) \to r) \to r).$$

They proved that $\mathbf{IBCWK_1X_3}$ is $\mathbf{TV}_\to$ and $\mathbf{BCKX_2} \neq \mathbf{BCKX_3}$.

However there is no proof that formula **I** is independent from **B, C, W, K$_1$, X$_3$**. So, let us replace formula $\mathbf{X_3}$ with $\mathbf{X_4}$:

$$\mathbf{X_4}.\ (p \to p) \to \mathbf{X_3}.$$

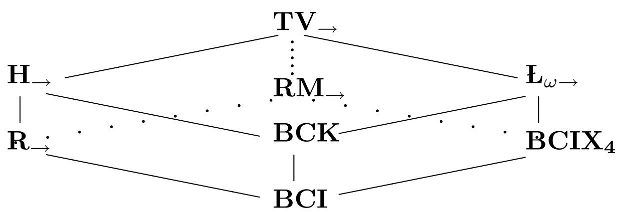
Theorem 4. *The set of formulas* **I, B, C, W, K$_1$, X$_4$** *is an independent axiomatization of* $\mathbf{TV}_\to$.

The special interest are $\mathbf{IBCWX_4}$ and $\mathbf{IBCK_1X_4}$ logics. It is a simple matter to prove that $\mathbf{IBCWX_4}$ is $\mathbf{RM}_\to$, where **IBCW** is $\mathbf{R}_\to$ (about $\mathbf{R}_\to$ and $\mathbf{RM}_\to$ see [1]).

But it is not trivial that $\mathbf{IBCK_1X_4}$ is $\mathbf{Ł}_{\omega\to}$ (see [6, pp. 158-160.], where $\mathbf{Ł}_{\omega\to}$ is an implicational fragment of Łukasiewicz's infinite-valued logic $\mathbf{Ł}_\omega$ [7]). So,

$$\mathbf{IBCK_1X_4} = \mathbf{IBCK_1X_3} = \mathbf{BCKX_3} = \mathbf{Ł}_{\omega\to}.$$

Note that $\mathbf{IBCK_1X_2}$ is only commutative **BCK** logic, i.e. **BCKD**. Now we can represent a lattice of implicational logics $L(\mathbf{TV}_\to)$ with axiom $\mathbf{X_4}$:



Pay attention to the very important result due to A.Avron [2]: $\mathbf{TV}_{\rightarrow}$ is a single proper extension of $\mathbf{RM}_{\rightarrow}$. In virtue of this property such lattice is called *maximal*.

## References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