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QUANTIFICATION IN INTUITIONISTIC LOGIC WITH PROVABILITY SMACK

A modified quantifier extension Q^+HC of the Heyting propositional calculus HC presented here is inspired, on the one hand, by the provability interpretation of the Intuitionistic logic (via Gödel's modal translation and Solovay's arithmetical completeness theorem). On the other hand, by the investigation [1], demonstrating how Arbitrary Objects lead naturally to the correct constraints on the rules of universal generalization and existential instantiation. On the last page of this book K. Fine points out: "In the light of this breakdown, logicians have experienced some difficulty in setting up a resonable system for intuitionistic logic with a rule of existential instantiation (see [2] and the references contained therein for details)".

We now present (and try to justify) an amendment to the standard quantifier extension QHC of the Heyting propositional calculus HC. Namely, our Amended Calculus Q^+HC is obtained from the usual QHC by postulating the following modified version of the rule of universal generalization

$$(+) \qquad \frac{\vdash (p(a) \to \forall x p(x)) \to p(a)}{\vdash \forall x p(x)} \; .$$

An alternative definition is expressed by

STATEMENT 1. *The calculus Q^+HC is equivalent to the calculus obtained from the usual QHC by accepting as an additional axiom the formula (Casari's schema)*

$$(Cas) \qquad \forall x[(p(x) \to \forall x p(x)) \to \forall x p(x)] \to \forall x p(x).$$

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Let us denote by $QHC^{\triangle}$ the standard quantifier extension of the Proof-Intuitionistic calculus $HC^{\triangle}$ of Kuznetsov-Muravitsky (see, for example, Kuznetsov [3]). The operator $^{\triangle}$ is intended to mean "is provable in Peano Arithmetic".

Statement 2. *The formula Cas is provable in the Calculus $QHC^{\triangle}$ (and, hence, admits a provability interpretation).*

Note that in the Calculus $QHC^{\triangle}$ the Rule (+) is expressible as follows

$$(++) \qquad \frac{\vdash \triangle\forall x p(x) \to p(a)}{\vdash \forall x p(x)} .$$

The calculus $Q^{+}HC$ is conservative over HC, i.e. provability of a quantifier-free formula in $Q^{+}HC$ implies its provability in HC. Disjunction and existential quantifier in $Q^{+}HC$ are constructive.

For every predicate formula p, denote by $e(p)$ (= embrace of p) the formula $\vdash e^{n+1}(p) \leftrightarrow e^{n}(p)$. Call a formula p cyclic if QHC $\vdash e^{n+1}(p) \leftrightarrow e^{n}(p)$ for some natural number n, where $e^{n+1}(p) = e(e^{n}(p))$.

The following statement clarifies status of our amendment.

Statement 3. *The logic $Q^{+}HC$ does not admit "nontrivial" cyclic formulæ, i. e. if for some n, $Q^{+}HC \vdash e^{n+1}(p) \leftrightarrow e^{n}(p)$, then $Q^{+}HC \vdash p$.*

Recall the remark of Heyting [4, p.104] in connection with the formula $\neg\neg\forall x p(x) \to \forall x \neg\neg p(x)$: "It is one of the most striking features of intuitionistic logic that the inverse implication does not hold, especially because the formula of the propositional calculus which results if we restrict x to a finite set, is true". And further: "It has been conjectured [5, p.46] that the formula $\forall x \neg\neg p(x) \to \neg\neg\forall x p(x)$ is always true if x ranges over a denumerable infinite species, but no way of proving the conjecture presents itself at present".

Statement 4. *The Kuroda formula (Kur) $\forall x \neg\neg p(x) \to \neg\neg\forall x p(x)$ and, consequently, also the biconditional $\forall x \neg\neg p(x) \leftrightarrow \neg\neg\forall x p(x)$ is provable in $Q^{+}HC$.*

Using figurative style, we note that there is a curious analogy with the state of affairs in Modal logics above S4. Namely, the formula (Kur) plays the role of the McKisey formula (McK), the well-known formula of

Dummett (Dum) behaves like the Minari formula

$$(Min) \qquad \neg\forall xp \vee \{\forall x[(p \to \forall xp) \to \forall xp] \to \forall xp\},$$

whereas the known connection Grz = Dum + McK "transforms" into Cas = Min + Kur.

In conclusion, let us note some semantic features of the Amended Intuitionistic logic: 1. Quantifier models of Q^+HC include all Kripke models with well-founded base (W, R) and, hence, all finitary, i.e. the ones with finite base (W, R). 2. Of the sheaf models, the logic Q^+HC "admits" the Sheaf toposes only over scattered Cantor spaces and hence also over ordinal spaces.

References

[1] K. Fine, **Reasoning with Arbitrary Objects**, Oxford, 1985.

[2] V. Smirnov, *Theory of Quantification and ε-calculi*, [in:] **Essay on Mathematical and Philosophical Logic** (J.Hintikka et al., eds), Reidel: Dordrecht-Holland, 1976, pp.41-47

[3] A. Kuznetsov, *On proof-intuitionistic calculus*, **Acad. Sci. USSR Doklady**, 1985, T. 283, pp. 27-29. (in Russian).

[4] A. Heyting, **Intuitionism**, North-Holland Publ. Comp., Amsterdam, 1956.

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