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AN ELEMENTARY PROOF OF EQUIVALLENCE OF CONDITIONS IN DEFINING OF CONDITIONALLY DISTRIBUTIVE LATTICES

In [2] B. Wolniewicz has defined the notion of *a conditionally distributive lattice* as such lattice $\mathcal{L} = < L, \sqcup, \sqcap, 1 >$ with the unit element 1, which satisfies two sentences:

(1) $\forall_{x,y,z \in L}[y \sqcup z \neq 1 \Rightarrow x \sqcap (y \sqcup z) = (x \sqcap y) \sqcup (x \sqcap z)]$,
(2) $\forall_{x,y,z \in L}[x \sqcup y \neq 1 \;\&\; x \sqcup z \neq 1 \Rightarrow x \sqcup (y \sqcap z) = (x \sqcup y) \sqcap (x \sqcup z)]$.

In [1] J. Hawranek and J. Zygmunt have given the criterion for conditionally distributive lattices, an analogue of Birkhoff's criterion for distributive lattices (see [1], s. 69–70). The criterion of Hawranek and Zygmunt results in equivalence of conditions (1) and (2) in each lattice with the unit element (see Corollary in [1]). In this paper we shall give an elementary proof of this equivalency.

Let $\mathcal{L} = < L, \sqcup, \sqcap, 1 >$ be an arbitrary lattice with the unit element. Let us recall that for all $x, y, z \in L$ we have:

(3)	$x \sqcup y$	$=$	$y \sqcup x$,
(4)	$x \sqcap y$	$=$	$y \sqcap x$,
(5)	$x \sqcup (y \sqcup z)$	$=$	$(x \sqcup y) \sqcup z$,
(6)	$x \sqcap (y \sqcap z)$	$=$	$(x \sqcap y) \sqcap z$,
(7)	x	$=$	$x \sqcup x$,
(8)	x	$=$	$x \sqcap x$,
(9)	x	$=$	$x \sqcup (x \sqcap y)$,
(10)	x	$=$	$x \sqcap (x \sqcup y)$,
(11)	$(x \sqcap y) \sqcup z$	$=$	$[(x \sqcap y) \sqcup z] \sqcap (y \sqcup z)$,
(12)	1	$=$	$1 \sqcup x$,
(13)	x	$=$	$1 \sqcap x$,
(14)	$x \sqcap y = 1$	$\Rightarrow$	$x = 1 = y$.

In an elementary way we shall prove that sentences (1) and (2) have in $\mathcal{L}$ the same logical value.

FACT 1. *If (1) is true in $\mathcal{L}$, then (2) is true in $\mathcal{L}$ too.*

PROOF. Let us assume that the condition (1) is true in $\mathcal{L}$ and take arbitrary $a, b, c \in L$ such that $a \sqcup b \neq 1$ and $a \sqcup c \neq 1$. Hence: $(a \sqcup b) \sqcap (a \sqcup c) \overset{(1)}{=} ((a \sqcup b) \sqcap a) \sqcup ((a \sqcup b) \sqcap c) \overset{(4,10)}{=} a \sqcup ((a \sqcup b) \sqcap c) \overset{(4,1)}{=} a \sqcup ((a \sqcap c) \sqcup (b \sqcap c)) \overset{(5,9)}{=} a \sqcup (b \sqcap c)$.

From (13) we get the following:

LEMMA 1. *For all $a, b, c \in L$ we have:* $1 \sqcap (b \sqcup c) = (1 \sqcap b) \sqcup (1 \sqcap c)$. □

Let us prove an auxiliary fact:

LEMMA 2. *If (2) is true in $\mathcal{L}$, the following sentence is also true in $\mathcal{L}$:*
(1') $\forall_{x,y,z \in L}(y \sqcup z \neq 1 \;\&\; (x \sqcup y \neq 1 \vee x \sqcup z \neq 1) \Rightarrow$
$x \sqcap (y \sqcup z) = (x \sqcap y) \sqcup (x \sqcap z))$

PROOF. Suppose that sentence (2) is true in $\mathcal{L}$ and let us take arbitrary $a, b, c \in L$ such that
(†) $b \sqcup c \neq 1$,
(‡) $a \sqcup b \neq 1$ or $a \sqcup c \neq 1$.
From (†), (11) and (14) it follows:
(††) $(a \sqcap b) \sqcup c \neq 1$ and $(a \sqcap c) \sqcup b \neq 1$.
Using Lemma 1 we can assume that
(*) $a \neq 1$.
Thus, by (9) and (3), we have
(**) $(a \sqcap b) \sqcup a \neq 1$ and $(a \sqcap c) \sqcup a \neq 1$.

If $a \sqcup c \neq 1$, then: $(a \sqcap b) \sqcup (a \sqcap c) \overset{(2,**,††)}{=} ((a \sqcap b) \sqcup a) \sqcap ((a \sqcap b) \sqcup c) \overset{(3,9)}{=} a \sqcap ((a \sqcap b) \sqcup c) \overset{(2,3,†,‡)}{=} a \sqcap ((a \sqcup c) \sqcap (b \sqcup c)) \overset{(6,10)}{=} a \sqcap (b \sqcup c)$. If $a \sqcup b \neq 1$, then similarly we can show that $(a \sqcap c) \sqcup (a \sqcap b) = a \sqcap (c \sqcup b)$, so it is enough if we use (3). □

FACT 2. *If (2) is true in $\mathcal{L}$, then (1) is true in $\mathcal{L}$ too.*

PROOF. Assume on the contrary that sentence (2) is true and (1) is false in $\mathcal{L}$. So there are some $a, b, c \in L$ that (†) holds true and $a \sqcap (b \sqcup c) \neq (a \sqcap b) \sqcup (a \sqcap c)$. Since $(a \sqcap b) \sqcup (a \sqcap c) \leq a \sqcap (b \sqcup c)$, then
(††) $(a \sqcap b) \sqcup (a \sqcap c) < a \sqcap (b \sqcup c)$.

By Lemma 1 let us suppose that (*) holds. Now we prove that

($) $(a \sqcap b) \sqcup c < [a \sqcap (b \sqcup c)] \sqcup c$ or $(a \sqcap c) \sqcup b < [a \sqcap (b \sqcup c)] \sqcup b$.

Indeed, using (†), (9), (5) and (3) we have $(a \sqcap b) \sqcup (a \sqcap c) \sqcup c \neq 1$ and $(a \sqcap b) \sqcup (a \sqcap c) \sqcup b \neq 1$. Therefore, by (2), we get: $(a \sqcap b) \sqcup (a \sqcap c) \sqcup (b \sqcap c) = [(a \sqcap b) \sqcup (a \sqcap c) \sqcup b] \sqcap [(a \sqcap b) \sqcup (a \sqcap c) \sqcup c] = [(a \sqcap c) \sqcup b] \sqcap [(a \sqcap b) \sqcup c]$. Assume towards contradiction that ($) is not true. So, by $(a \sqcap b) \sqcup c \leq [a \sqcap (b \sqcup c)] \sqcup c$ and $(a \sqcap c) \sqcup b \leq [a \sqcap (b \sqcup c)] \sqcup b$, we have

$(a \sqcap c) \sqcup b = [a \sqcap (b \sqcup c)] \sqcup b$ and $(a \sqcap b) \sqcup c = [a \sqcap (b \sqcup c)] \sqcup c$.

Then: $(a \sqcap b) \sqcup (a \sqcap c) \sqcup (b \sqcap c) = \{[a \sqcap (b \sqcup c)] \sqcup b\} \sqcap \{[a \sqcap (b \sqcup c)] \sqcup c\}$. From (†) it follows, that $[a \sqcap (b \sqcup c) \sqcup b \neq 1$ and $[a \sqcap (b \sqcup c)] \sqcup c \neq 1$. Using (2), we get: $(a \sqcap b) \sqcup (a \sqcap c) \sqcup (b \sqcap c) = [a \sqcap (b \sqcup c)] \sqcup (b \sqcap c)$. By (*), (9) and (††), using (2) we have: $(a \sqcap c) \sqcup (a \sqcap b) = [(a \sqcap c) \sqcup a] \sqcap (a \sqcap c) \sqcup b] = a \sqcap [(a \sqcap c) \sqcup b]$. Similarly we can show: $(a \sqcap b) \sqcup (a \sqcap c) = a \sqcap [(a \sqcap b) \sqcup c]$. From these identities, by (8) and (3), we get: $(a \sqcap b) \sqcup (a \sqcap c) = \{a \sqcap [(a \sqcap b) \sqcup c]\} \sqcap \{a \sqcap [(a \sqcap c) \sqcup b]\}$. Now using (4), (6) and (8) we get: $(a \sqcap b) \sqcup (a \sqcap c) = a \sqcap [(a \sqcap c) \sqcup b] \sqcap [(a \sqcap b) \sqcup c]$. So form proven identities we get $(a \sqcap b) \sqcup (a \sqcap c) = a \sqcap [(a \sqcap b) \sqcup (a \sqcap c) \sqcup (b \sqcap c)]$, and next $(a \sqcap b) \sqcup (a \sqcap c) = a \sqcap \{[a \sqcap (b \sqcup c)] \sqcup (b \sqcap c)\}$. Thus, since $a \sqcap (b \sqcup c) \leq a$ and $a \sqcap (b \sqcup c) \leq [a \sqcap (b \sqcup c)] \sqcup (b \sqcap c)$, so $a \sqcap (b \sqcup c) \leq a \sqcap \{[a \sqcap (b \sqcup c)] \sqcup (b \sqcap c)\} = (a \sqcap b) \sqcup (a \sqcap c)$ too. This contradicts (††), and thus ends the proof of condition ($).

Now assume that the first element of disjunction ($) is fulfilled.

Because $(a \sqcap b) \sqcup c \leq c \sqcup (a \sqcap (b \sqcup c)) \leq b \sqcup c$ and $b \sqcup c = [(a \sqcap b) \sqcup c] \sqcup [(a \sqcap c) \sqcup b]$, thus by (†) and (2) one can get: $\{[(a \sqcap b) \sqcup c] \sqcup [c \sqcup (a \sqcap (b \sqcup c))]\} \sqcap \{[(a \sqcap b) \sqcup c] \sqcup [(a \sqcap c) \sqcup b]\} = [(a \sqcap b) \sqcup c] \sqcup \{[c \sqcup (a \sqcap (b \sqcup c))] \sqcap [(a \sqcap c) \sqcup b]\}$. Since $c \sqcup (a \sqcap (b \sqcup c)) = \{[(a \sqcap b) \sqcup c] \sqcup [c \sqcup (a \sqcap (b \sqcup c))]\} \sqcap \{[(a \sqcap b) \sqcup c] \sqcup [(a \sqcap c) \sqcup b]\}$, consequently

$c \sqcup (a \sqcap (b \sqcup c)) = [(a \sqcap b) \sqcup c] \sqcup \{c \sqcup (a \sqcap (b \sqcup c))] \sqcap [(a \sqcap c) \sqcup b]\}$.

Since $1 \neq b \sqcup c = c \sqcup [(a \sqcap c) \sqcup b]$, using (1') we get:

$[c \sqcup (a \sqcap (b \sqcup c))] \sqcap [(a \sqcap c) \sqcup b] = \{c \sqcap [(a \sqcap c) \sqcup b]\} \sqcup \{[a \sqcap [(b \sqcup c)] \sqcap [(a \sqcap c) \sqcup b]\}$.

After some transformations one can get:

$[c \sqcup (a \sqcap (b \sqcup c))] \sqcap [(a \sqcap c) \sqcup b] = \{c \sqcap [(a \sqcap c) \sqcup b]\} \sqcup \{a \sqcap [(a \sqcap c) \sqcup b]\}$.

Since, by (†), we have $c \neq 1$ and $(a \sqcap c) \sqcup b \neq 1$, thus using (2) i (9) we get: $c \sqcap [(a \sqcap c) \sqcup b] = [(a \sqcap c) \sqcup c] \sqcap [(a \sqcap c) \sqcup b] = (a \sqcap c) \sqcup (b \sqcap c)$. Analogously, using (*), we have $a \sqcap [(a \sqcap c) \sqcup b] = (a \sqcap c) \sqcup (a \sqcap b)$. Therefore

$[c \sqcup (a \sqcap (b \sqcup c))] \sqcap [(a \sqcap c) \sqcup b] = (a \sqcap c) \sqcup (a \sqcap b) \sqcup (b \sqcap c)$.

By earlier considerations we get:

$$c \sqcup (a \sqcap (b \sqcup c)) = [(a \sqcap b) \sqcup c] \sqcup (a \sqcap c) \sqcup (a \sqcap b) \sqcup (b \sqcap c).$$

And so, by (7) and (9), we have

$$c \sqcup (a \sqcap (b \sqcup c)) = (a \sqcap b) \sqcup c,$$

which contradicts the first element of disjunction ($).

In an analogous way we get a contradiction when we assume the second element of the disjunction ($). So in both cases we get contradictions. □

References

[1] J. Hawranek and J. Zygmunt, *Some elementary properties of conditionally distributive lattices*, **Bulletin of the Section of Logic** 12/3 (1983), pp. 117–125.

[2] B. Wolniewicz, *On the lattice of elementary situation*, **Bulletin of the Section of Logic** 9/3 (1980), pp. 115–121.

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