Professor Rasiowa [HR49] considers implication algebras \((A, \Rightarrow, V)\) such that \(\Rightarrow\) is a binary operation on the universe \(A\) and \(V \in A\). In particular, there are studied \textit{implicative algebras}, \textit{positive implication algebras} and \textit{implication algebras}; the second kind corresponds to the logic of positive implication, and the third one to the logic of classical implication. In each case, the distinguished element \(V\) is interpreted as the equivalence class of all theorems in the appropriate Lindenbaum algebra. Accordingly, the approach cannot be directly applied to logics in which theorems do not form a single equivalence class (e.g. BCI) or there are no theorems at all (e.g. the Lambek calculus); such logos are typical for the world of substructural logics, arising from Gentzen style systems of intuitionistic logic by dropping structural rules (see [3]). Instead of implication algebras in the sense of [HR49], one should consider \textit{implication structures} \((A, \Rightarrow, \leq)\) such that \((A, \leq)\) is a poset, and \(\Rightarrow\) is a binary operation on \(A\). In the Lindenbaum algebra, \(\leq\) corresponds to the (sequential) consequence relation \(\vdash\), determined by the given logic.

Usually, implication structures are obtained as reducts of \textit{residuated groupoids} \((A, \circ, \Rightarrow, \Leftarrow, \leq)\), where \((A, \leq)\) is a poset, and \(\circ, \Rightarrow, \Leftarrow\) are binary operations on \(A\), fulfilling the equivalences:

\[
\text{(RG) } b \leq a \Rightarrow c \iff a \circ b \leq c \iff a \leq c \Leftarrow b,
\]

for all \(a, b, c \in A\). Notice that (RG) mirrors introduction and elimination rules for implication in the Natural Deduction format; \(\circ\) is the \textit{product} (or: fusion) operation which corresponds to joining formulas in the antecedent of a sequent. If the order of formulas in the antecedent of a sequent is irrelevant, then \(\circ\) is supposed to be commutative, and consequently, \(\Rightarrow = \Leftarrow\) corresponds to the logical constant \(\rightarrow\). Noncommutative structures are related to noncommutative logics with two conditionals \(\rightarrow\) and \(\leftarrow\). If \(\circ\) is associative (antecedents of sequents are strings of formulas), then the
structure above is called a residuated semigroup. If additionally $1 \in A$ is the unit for $\circ$, then this structure is called a residuated monoid; residuated monoids correspond to logics with theorems (provable sequents with empty antecedents). Observe that each residuated groupoid satisfies the monotonicity conditions:

(M1) if $a \leq b$, then $c \circ a \leq c \circ b$ and $a \circ c \leq b \circ c$,
(M2) if $a \leq b$, then $c \Rightarrow a \leq c \Rightarrow b$ and $b \Rightarrow c \leq a \Rightarrow c$,
(M3) if $a \leq b$, then $a \Leftarrow c \leq b \Leftarrow c$ and $c \Leftarrow b \leq c \Leftarrow a$.

We shall consider residuated algebras of sets: concrete residuated algebras in the sense of Dunn [4] and powerset residuated algebras in the sense of [2]. The former come from Kripke style frames for relevant logics. A ternary frame is a pair $(U, R)$ such that $U$ is a nonempty set, and $R \subseteq U^3$. For $X, Y \subseteq U$, we define:

\[
X \circ Y = \{ z \in U : (\exists x \in X)(\exists y \in Y) R(x, y, z) \}, \\
X \Rightarrow Y = \{ y \in U : (\forall x \in X)(\text{if } x \in X \text{ and } R(x, y, z) \text{ then } z \in Y) \}, \\
X \Leftarrow Y = \{ x \in U : (\forall y \in Y)(\text{if } y \in Y \text{ and } R(x, y, z) \text{ then } z \in X) \}.
\]

The structure $(P(U), \circ, \Rightarrow, \Leftarrow, \subseteq)$ is a residuated groupoid, called the concrete residuated groupoid over $(U, R)$. If $R$ fulfils the condition:

\[
(\exists z)(R(x, y, z) \land R(z, u, w)) \iff (\exists v)(R(y, u, v) \land R(x, v, w)),
\]

then the concrete residuated groupoid over $(U, R)$ is a residuated semigroup. Other constraints upon $R$ yield commutative residuated semigroups, groupoids, monoids, etc.

More special structures are based on groupoids $(U, \cdot)$ which give rise to ternary frames $(U, R)$ such that $R(x, y, z)$ holds iff $x \cdot y = z$; we call them powerset residuated groupoids over groupoids $(U, \cdot)$. Notice that, in the powerset residuated groupoid, $\circ$ is the standard powerset operation determined by:

\[
X \circ Y = \{ x \cdot y : x \in X, y \in Y \},
\]

while $\Rightarrow, \Leftarrow$ are given by:

\[
X \Rightarrow Y = \{ y \in U : (\forall x \in X) x \cdot y \in Y \}, \\
X \Leftarrow Y = \{ x \in U : (\forall y \in Y) x \cdot y \in X \}.
\]
Clearly, if \((U, \cdot)\) is a (resp. commutative) semigroup, then \(P(U)\) is a (resp. commutative) residuated semigroup, and if \((U, \cdot, 1)\) is a (resp. commutative) monoid, then \((P(U), \ldots, \{1\})\) is a (resp. commutative) residuated monoid. We refer to these structures as (resp. commutative) powerset residuated semigroups and monoids.

Representation theorems to be discussed here are of the general form: each implication structure (residuated algebra) of the given kind is isomorphic to a substructure of some concrete or powerset residuated algebra.

Representation in concrete residuated algebras can be accomplished by methods analogous to those used in [HR49] for implication algebras (which follow the well known Stone representation theorem for Boolean algebras). We give a typical example (after Dunn [4]).

Let \((A, \circ, \Rightarrow, \Leftarrow, \leq)\) be a residuated groupoid. We show that this structure is isomorphic to a substructure of a concrete residuated groupoid. By a cone we mean a set \(\nabla \subseteq A\) such that, for all \(x, y \in A\), if \(x \in \nabla\) and \(x \leq y\), then \(y \in \nabla\). Let \(U\) be the set of all cones, and let \(R \subseteq U^3\) be defined by:

\[
R(\nabla_1, \nabla_2, \nabla_3) \text{ iff } (\forall x \in \nabla_1)(\forall y \in \nabla_2) x \circ y \in \nabla_3.
\]

As above, we construct the concrete residuated groupoid \(P(U)\). One shows that the mapping:

\[
h(x) = \{\nabla \in U : x \in \nabla\}
\]

is a monomorphism of the residuated groupoid \((A, \ldots)\) into the concrete residuated groupoid \(P(U)\) which fulfils the equivalence:

\[
x \leq y \text{ iff } h(x) \subseteq h(y), \text{ for all } x, y \in A.
\]

Consequently, each residuated groupoid is embeddable into a concrete residuated groupoid. Quite analogously, one proves other theorems of that kind, for instance:

(A) each (commutative) residuated semigroup (monoid) is embeddable into a (commutative) concrete residuated semigroup (monoid),

(B) each implication structure \((A, \Rightarrow, \Leftarrow, \leq)\), fulfilling \((M2), (M3)\) and the laws:

\[
a \leq b \Leftarrow (a \Rightarrow b), \ a \leq (b \Leftarrow a) \Rightarrow b
\]

is embeddable into a concrete residuated groupoid.
(C) each implication structure \((A, \Rightarrow, \leq)\), fulfilling (M2), is embeddable into a concrete residuated groupoid.

For (B) and (C), one defines ternary relations:

\[
R_{\Rightarrow}(\nabla_1, \nabla_2, \nabla_3) \text{ iff } (\forall x \in \nabla_1)((\forall y \in \nabla_2) (\text{if } x \Rightarrow y \in \nabla_2 \text{ then } y \in \nabla_3),
\]

\[
R_{\Leftarrow}(\nabla_1, \nabla_2, \nabla_3) \text{ iff } (\forall y \in \nabla_2)((\forall x \in \nabla_1) (\text{if } x \Leftarrow y \in \nabla_1 \text{ then } x \in \nabla_3).
\]

The additional laws in (B) yield \(R_{\Rightarrow} = R_{\Leftarrow}\), and the monomorphism \(h\) is defined as above.

Representation in powerset residuated algebras can be accomplished by the above methods only for implication structures but not for structures with \(\circ\). Again, let \((A, \Rightarrow, \leq)\) be an implication structure. Let \(U\) be defined as above. The operation \(\cdot\) on \(U\) is given by:

\[
\nabla_1 \cdot \nabla_2 = \{ y \in A : (\exists x \in \nabla_1) x \Rightarrow y \in \nabla_2 \}.
\]

We construct the powerset residuated groupoid \(P(U)\) and define \(h\) as above. If \((A, \Rightarrow, \leq)\) fulfills (M2), then \(h\) is a monomorphism from \((A, \Rightarrow, \leq)\) to the \(\Rightarrow, \subseteq\)-reduct of \(P(U)\). Also, if \((A, \Rightarrow, \Leftarrow, \leq)\) fulfills the conditions from (B), then \(h\) is a monomorphism from the latter structure to the \(\Rightarrow, \Leftarrow, \subseteq\)-reduct of \(P(U)\). However, \(h\) is not a homomorphism from the residuated groupoid \((A, \circ, \Rightarrow, \Leftarrow, \leq)\) to \(P(U)\); we only have \(h(x) \circ h(y) \subseteq h(x \circ y)\), while the converse inclusion fails, in general.

The following representation theorem:

\((\text{RT})\) each residuated semigroup is embeddable into a powerset residuated semigroup

and similar results for residuated groupoids, monoids, commutative structures etc. can be proven with the aid of proof-theoretic tools from [1]. We use the Lambek calculus \(L\) [7] which deals with sequents \(A \vdash B\), where \(A, B\) are propositional formulas with logical constants \(\circ, \Rightarrow, \Leftarrow\). The axioms and rules of \(L\) are:

\[(\text{A0}) A \vdash A,\]

\[
(\text{A1}) (A \circ B) \circ C \vdash A \circ (B \circ C), \quad (\text{A2}) A \circ (B \circ C) \vdash (A \circ B) \circ C,\]

\[
(\text{R1}) A \circ B \vdash C, \quad (\text{R2}) A \circ B \vdash C \quad B \vdash A \Rightarrow C, \quad A \vdash C \Leftarrow B.
\]
If \( \Phi \) is a set of sequents, then \( L(\Phi) \) denotes the system \( L \) enriched by all sequents from \( \Phi \) as new axioms. It is easy to show that sequents derivable in \( L \) are precisely those which are valid in all residuated semigroups (that means, true under all valuations \( \alpha \); \( A \vdash B \) is true under \( \alpha \), if \( \alpha(A) \leq \alpha(B) \)). Further, sequents derivable in \( L(\Phi) \) are precisely those which are true in all residuated semigroups under all valuations \( \alpha \) which satisfy all sequents from \( \Phi \). Thus, \( L \) is strongly complete with respect to the semantics of residuated semigroups (use Lindenbaum algebras).

We construct a Labelled Deductive System (in the sense of GABBA [5]), denoted by LDS, which is a conservative extension of \( L(\Phi) \). Labels are defined as follows: (i) all formulas are labels, (ii) if \( s \) and \( t \) are labels, then \( st \) is a label, (iii) if \( s \) is a label, and \( A, B \) are formulas, then \( (s, 1, A \circ B) \) and \( (s, 2, A \circ B) \) are labels. LDS deals with labelled formulas \( s : A \) whose intended meaning is: the element \( s \in M \) belongs to the set \( \alpha(A) \subseteq M \).

The axioms and rules of LDS are:

\[
(A) \quad s : A, \quad \frac{s : A \circ B}{st : A \circ B} \quad (E_\circ 1) \quad \frac{s : A \circ B \quad (s, 1, A \circ B) : A'}{s : A \circ B \quad (s, 2, A \circ B) : B'}, \quad (E_\circ 2)
\]

\[
(I\rightarrow) \quad s : A ; t : B \quad (E\rightarrow) \quad \frac{As : B}{st : B}, \quad (s : A \rightarrow B) : B', \quad (s : A \rightarrow B) : B'
\]

\[
(I\leftarrow) \quad sA : B \quad (E\leftarrow) \quad \frac{s : B \leftarrow A \quad st : B}{s : B \leftarrow A \quad st : B, \quad (s : B \leftarrow A) : A'}
\]

\[
(R) \quad \frac{s : A}{As : B}, \quad \frac{s : A}{t : A}, \quad \text{if } s \text{ reduces to } t,
\]

\[
(R\Phi) \quad \frac{s : A}{s : B}, \quad \text{if } A \vdash B \text{ is in } \Phi.
\]

In rule (R), \( s \) reduces to \( t \), if \( t \) arises from \( s \) by a finite number of replacements of a redex \( (u, 1, B \circ C)(u, 2, B \circ C) \) with its contractum \( u \). For every term \( s \), there is precisely one irreducible term \( s^* \) such that \( s \) reduces to \( s^* \).

Let \( M \) consist of all irreducible terms. We consider the semigroup \((M, \cdot,\cdot)\),
where, for \( s, t \in M \), \( s \cdot t = (st)^* \). The powerset model \((P(M), \alpha)\) such that 
\[
\alpha(p) = \{ s \in M : \text{LDS } \vdash s : p \}
\]
satisfies the canonical equivalence:

\[
(P(M), \alpha) \models A \vdash B \iff \text{L(\Phi) } \vdash A : B.
\]

Further, LDS is conservative over \( \text{L(\Phi)} \): \( A \vdash B \) is derivable in \( \text{L(\Phi)} \) iff \( A : B \) is derivable in LDS. That immediately yields the strong completeness of \( \text{L} \) with respect to powerset residuated semigroups.

We prove the representation theorem (RT). Let \((A, \circ, \Rightarrow, \Leftarrow, \leq)\) be a residuated semigroup. To each \( a \in A \) we assign a different atomic formula \( p_a \); we assume all atomic formulae are of that form. An assignment \( \mu \) in \((A, \ldots)\) is given by \( \mu(p_a) = a \). Let \( \Phi \) consist of all sequents \( A \vdash B \) which are true in the model \((A, \mu)\). We construct the canonical model \((P(M), \alpha)\) for \( \text{L(\Phi)} \), as above. It is easy to show that the mapping \( h : A \mapsto P(M) \), defined by \( h(a) = \alpha(p_a) \) is a monomorphism of \((A, \ldots)\) into \((P(M), \ldots)\).

Analogous theorems can be proven for residuated groupoids, monoids etc. In \([6]\), similar results have been obtained for abstract residuated algebras. We can also consider richer structures with additional operations, e.g. lattice operations and negation, and we can prove representation theorems either by the methods of \([HR49]\) (for structures without \( \circ \)), or by the methods akin to the above.

Let us mention an open question. According to Bialynicki-Birula and Rasiowa \([HR18]\), a De Morgan negation in the poset \((A, \leq)\) is a unary operation \( \neg \) on \( A \) such that there hold the laws of Double Negation \( \neg\neg a = a \) and Transposition: \( a \leq b \) implies \( \neg b \leq \neg a \). For \( X \subseteq U \), the quasi-Boolean complement of \( X \) is defined by: \( \neg X = U - f[X] \), where \( f \) is a fixed mapping from \( U \) onto itself such that \( f(f(x)) = x \), for all \( x \in U \). One can prove that each residuated groupoid (semigroup) with De Morgan negation is embeddable into a concrete residuated groupoid (semigroup) with quasi-Boolean complement (an analogue of the representation theorem for quasi-Boolean algebras in \([HR18]\)). In the universe \( U \), of all cones, define \( f(\nabla) = A - \neg(\nabla) \), where \( \neg(\nabla) \) is the set of all \( \neg(a) \), for \( a \in \nabla \). In a similar way, one can prove that the \( \circ \)-free reduct of any residuated groupoid (semigroup) with De Morgan negation is embeddable into a powerset residuated groupoid with quasi-Boolean complement. We do not know if each residuated groupoid (semigroup) with De Morgan negation is embeddable into a powerset residuated groupoid with quasi-Boolean complement.
References


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