A GENTZEN SYSTEM EQUIVALENT TO THE BCK-LOGIC

Abstract
The sequent calculus $L_{BCK}$ is obtained by deleting the contraction rule and the introduction rules of the connectives of meet, join and negation from the sequent calculus for the Intuitionistic Propositional logic, $L_I$. In this paper we show that the Gentzen system $G_{BCK}$, naturally associated with $L_{BCK}$, is equivalent to the Hilbert-style logic $BCK$, in the sense that $G_{BCK}$ and $BCK$ are interpretable one another, the interpretations being essentially inverse in each other.

This result is a strengthening of the one obtained by H. Ono and Y. Komori [8], which concerned only the derivable sequents of $L_{BCK}$ and the theorems of $BCK$.

We obtain, as a corollary of this result, that the quasivariety of $BCK$-algebras is the equivalent quasivariety semantics of $G_{BCK}$ in the sense of [10].

Let $\mathcal{L}$ be the propositional language with a single binary connective, $\rightarrow$. The logic $BCK$ ($(\mathcal{L}, \vdash_{BCK})$, for short) is the 1-dimensional deductive system over $\mathcal{L}$ defined by the axioms:

(B) $(X \rightarrow Y) \rightarrow ((Y \rightarrow Z) \rightarrow (X \rightarrow Z))$
(C) $(X \rightarrow (Y \rightarrow Z)) \rightarrow (Y \rightarrow (X \rightarrow Z))$
(K) $X \rightarrow (Y \rightarrow X)$

and the modus ponens rule: $\langle \{X, X \rightarrow Y\}, Y \rangle$, where $X, Y, Z \in Fm_\mathcal{L}$.

Recall that it is well known that the following formulas are theorems of $BCK$:

(I) $(X \rightarrow Y) \rightarrow ((Z \rightarrow X) \rightarrow (Z \rightarrow Y))$
(II) $(X_1 \rightarrow (..(X_n \rightarrow Y)\ldots)) \rightarrow ((Y \rightarrow Z) \rightarrow (X_1 \rightarrow (..(X_n \rightarrow Z)\ldots)))$

where $X_1, \ldots, X_n, X, Y, Z \in Fm_\mathcal{L}$.

Recall that the quasivariety $\mathcal{BCK}$ is the class of algebras of type $\{2\}$ which satisfy the following equations and quasiequations:

(1) $(X \rightarrow Y) \rightarrow ((Y \rightarrow Z) \rightarrow (X \rightarrow Z)) = X \rightarrow X$
(2) $X \rightarrow ((X \rightarrow Y) \rightarrow Y) = X \rightarrow X$

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Y → (X → X) = X → X
(4) X → Y = X → X \land Y → X = X → X \Rightarrow X = Y,
where X, Y, Z \in Fm_L.

It is known that the BCK is not a variety [11].

Next we introduce a Gentzen system \( G_{BCK} \), and we will show in Theorem 7 that it is equivalent to the deductive system \( BCK \).

**Definition 1.** Let \( L = \{ \to \} \) and let \( \Gamma, \Pi \) be finite sequences of \( L \)-formulas and \( \varphi, \psi, \xi \) \( L \)-formulas. The sequent calculus \( L_{BCK} \) is defined by the following axiom and rules:

\[
\varphi \vdash \varphi \quad \text{(Axiom)}
\]

\[
\frac{\Gamma \vdash \varphi, \Pi \vdash \xi}{\Gamma, \Pi \vdash \xi} \quad \text{(cut)}
\]

\[
\frac{\Gamma, \varphi, \psi, \Pi \vdash \xi}{\Gamma, \psi, \varphi, \Pi \vdash \xi} \quad \text{(e \vdash)}
\]

\[
\frac{\Gamma \vdash \xi}{\Gamma, \varphi \vdash \xi} \quad \text{(w \vdash)}
\]

\[
\frac{\Gamma \vdash \varphi, \psi, \Pi \vdash \xi}{\varphi \to \psi, \Gamma, \Pi \vdash \xi} \quad \text{(→ \vdash)}
\]

\[
\frac{\Gamma, \varphi \vdash \psi}{\Gamma \vdash \varphi \to \psi} \quad \text{(\vdash \to)}
\]

This sequent calculus was studied by H. Ono and Y. Komori [8]. They proved the cut elimination theorem and that this sequent calculus is logically equivalent to the \( BCK \)-logic, that is, for any formula \( \gamma_1, \ldots, \gamma_n, \varphi \), the sequent \( \gamma_1, \ldots, \gamma_n \vdash \varphi \) is derivable in \( L_{BCK} \) if and only if \( \gamma_1 \to \cdots \to (\gamma_n \to \varphi) \) is a theorem of the \( BCK \)-logic.

The sequent calculus \( L_{BCK} \) allows us to define the consequence relation \( \vdash_{L_{BCK}} \) on the set of sequents in the following way:

Let \( T \) be a set of sequents and \( \Gamma \vdash \varphi \) a sequent, then \( T \vdash_{L_{BCK}} \Gamma \vdash \varphi \) if and only if there is a finite sequence of sequents \( \Gamma_0 \vdash \varphi_0, \ldots, \Gamma_{n-1} \vdash \varphi_{n-1} \) (which is called a proof of \( \Gamma \vdash \varphi \) from \( T \)) such that \( \Gamma_{n-1} \vdash \varphi_{n-1} = \Gamma \vdash \varphi \) and for each \( i < n \) one of the following conditions holds:

(i) \( \Gamma_i \vdash \varphi_i \) is an instance of the axiom;
(ii) \( \Gamma_i \vdash \varphi_i \in T \);
(iii) \( \Gamma_i \vdash \varphi_i \) is obtained from \( \{ \Gamma_j \vdash \varphi_j : j < i \} \) by using a rule of \( L_{BCK} \).
We define the Gentzen system $\mathcal{G}_{BCK}$ as the pair $\langle \mathcal{L}, \vdash_{L_{BCK}} \rangle$.

**Definition 2.** Let $\Gamma$ be a finite sequence of $\mathcal{L}$-formulas and $\varphi$ an $\mathcal{L}$-formula.

We define a $(0,1)$-deductive system $S_{\mathcal{G}_{BCK}} = \langle \mathcal{L}, \vdash_{S_{\mathcal{G}_{BCK}}} \rangle$ associated to $\mathcal{G}_{BCK}$ as the deductive system given by:

\[ \Gamma \vdash_{S_{\mathcal{G}_{BCK}}} \varphi \iff \{ \emptyset \vdash \gamma : \gamma \in \Gamma \} \vdash_{L_{BCK}} \emptyset \vdash \varphi. \]

**Theorem 3.** Let $\tau$ and $\rho$ be the translations from $\mathcal{G}_{BCK}$ to $S_{\mathcal{G}_{BCK}}$ and from $S_{\mathcal{G}_{BCK}}$ to $\mathcal{G}_{BCK}$ respectively, defined by:

\[ \tau_{(m,1)}(p_0, \ldots, p_{m-1} \vdash q_0) = \begin{cases} p_0 \rightarrow (p_1 \rightarrow (\ldots \rightarrow (p_{m-1} \rightarrow q_0)\ldots)) & \text{if } m \neq 0 \\ q_0 & \text{if } m = 0 \end{cases} \]

\[ \rho(p_0) = \{ \emptyset \vdash p_0 \}. \]

Then we have:

(i) \( \{ \Gamma_i \vdash \gamma_i : i \in I \} \vdash_{L_{BCK}} \Gamma \vdash \gamma \iff \{ \tau (\Gamma_i \vdash \gamma_i) : i \in I \} \vdash_{S_{\mathcal{G}_{BCK}}} \tau (\Gamma \vdash \gamma). \)

(ii) \( \varphi \vdash \tau_{(0,1)}(\emptyset \vdash \phi). \)

**Proof.**

(a) It is easy to check property (i) by using that every sequent is equivalent, modulo $\mathcal{G}_{BCK}$, to a sequent with the empty sequence on the left of the turnstyle.

(b) It is easy to check property (ii) by using the definitions of $\tau$ and $\rho$. □

According to the definition of equivalence between systems given by J. Rebagliato and V. Verdú [10], the theorem shows that this $(0,1)$-deductive system $S_{\mathcal{G}_{BCK}}$ is equivalent to the Gentzen system $\mathcal{G}_{BCK}$.

Now we will show that $\vdash_{S_{\mathcal{G}_{BCK}}} = \vdash_{BCK}$. To prove this equality we will use the following lemmas.

**Lemma 4.** Let $\tau$ be the translation of $\mathcal{L}$-sequents in $\mathcal{L}$-formulas defined in

**Theorem 3.** Then for every rule of $L_{BCK}, \frac{\Delta \vdash \delta : j \leq 2}{\Delta \vdash \delta}$, we have

\[ \{ \tau (\Delta_j \vdash \delta_j) : j \leq 2 \} \vdash_{BCK} \tau (\Delta \vdash \delta). \]
PROOF. We distinguish cases:

- Cut rule: we have to show that
  \[ \{ \tau (\Gamma \vdash \varphi), \tau (\varphi, \Pi \vdash \xi) \} \vdash_{BCK} \tau (\Gamma, \Pi \vdash \xi) \]  
  \((\text{cutH})\).

  If \( \Gamma = \emptyset \) we have to show that \( \{ \varphi, \varphi \rightarrow \tau (\Pi \vdash \xi) \} \vdash_{BCK} \tau (\Pi \vdash \xi) \).

  By using modus ponens, the result follows.

  If \( \Gamma = \{ \gamma_1, ..., \gamma_n \} \) then \( \tau (\Gamma, \Pi \vdash \xi) = \gamma_1 \rightarrow (\ldots \rightarrow (\gamma_n \rightarrow (\tau (\Pi \vdash \xi)))) \).

  \( \tau (\varphi, \Pi \vdash \xi) = \varphi \rightarrow \tau (\Pi \vdash \xi) \), and \( \tau (\Gamma \vdash \varphi) = \gamma_1 \rightarrow (\ldots \rightarrow (\gamma_n \rightarrow \varphi)) \).

  By applying the property (II) of BCK, we obtain
  \( \emptyset \vdash_{BCK} \tau (\Gamma \vdash \varphi) \rightarrow (\tau (\varphi, \Pi \vdash \xi) \rightarrow \tau (\Gamma, \Pi \vdash \xi)) \).

  Now by applying modus ponens \( \{ \tau (\Gamma \vdash \varphi), \tau (\varphi, \Pi \vdash \xi) \} \vdash_{BCK} \tau (\Gamma, \Pi \vdash \xi) \).

- Exchange rule: we have to show that
  \( \tau (\Gamma, \varphi, \psi, \Pi \vdash \xi) \vdash_{BCK} \tau (\Gamma, \psi, \varphi, \Pi \vdash \xi) \) \((eH)\).

  If \( \Gamma = \emptyset \), then by using (C) we have: \( \emptyset \vdash_{BCK} \xi \rightarrow (\varphi \rightarrow \xi) \) and by applying modus ponens, the result follows.

  If \( \Gamma \neq \emptyset \), then from the last expression, the property (I) and modus ponens we obtain \( \tau (\Gamma, \varphi, \psi, \Pi \vdash \xi) \vdash_{BCK} \tau (\Gamma, \psi, \varphi, \Pi \vdash \xi) \).

- Weakening rule: we have to show that
  \( \{ \tau (\Gamma \vdash \xi) \} \vdash_{BCK} \tau (\Gamma \vdash \xi) \) \((wH)\).

  If \( \Gamma = \emptyset \), then by using (K) we have: \( \vdash_{BCK} \xi \rightarrow (\varphi \rightarrow \xi) \) and by applying modus ponens, we obtain \( \{ \tau (\emptyset \vdash \xi) \} \vdash_{BCK} \tau (\varphi \vdash \xi) \).

  If \( \Gamma \neq \emptyset \), then from the last expression and property (I) we obtain
  \( \{ \tau (\Gamma \vdash \xi) \} \vdash_{BCK} \tau (\Gamma, \varphi \vdash \xi) \).

- Left introduction rule for \( \rightarrow \): we have to show that
  \( \{ \tau (\Gamma \vdash \varphi), \tau (\psi, \Pi \vdash \xi) \} \vdash_{BCK} \tau (\varphi \rightarrow \psi, \Pi \vdash \xi) \) \((\rightarrow H)\).

  From property (I) we have:
  \( \emptyset \vdash_{BCK} (\psi \rightarrow (\varphi, \Pi \vdash \xi)) \rightarrow ((\varphi \rightarrow \psi) \rightarrow (\varphi \rightarrow (\Pi \vdash \xi))) \) and by applying modus ponens, we obtain \( \tau (\psi, \Pi \vdash \xi) \vdash_{BCK} \tau (\varphi \rightarrow \psi, \varphi, \Pi \vdash \xi) \).

  Now by using (eH) we obtain \( \tau (\psi, \Pi \vdash \xi) \vdash_{BCK} \tau (\varphi \rightarrow \psi, \varphi, \Pi \vdash \xi) \) and by applying \( (\text{cutH}) \): \( \{ \tau (\Gamma \vdash \varphi), \tau (\varphi, \varphi \rightarrow \psi, \Pi \vdash \xi) \} \vdash_{BCK} \tau (\Gamma, \varphi \rightarrow \psi, \Pi \vdash \xi) \).

  We obtain \( \{ \tau (\Gamma \vdash \varphi), \tau (\psi, \Pi \vdash \xi) \} \vdash_{BCK} \tau (\Gamma, \varphi \rightarrow \psi, \Pi \vdash \xi) \).

  Now by applying \( (\text{cutH}) \), we use \( \{ \tau (\Gamma \vdash \varphi), \tau (\varphi, \Pi \vdash \xi) \} \vdash_{BCK} \tau (\Gamma, \varphi \rightarrow \psi, \Pi \vdash \xi) \).

  By using (eH), the result \((\rightarrow H)\) follows.

- Right introduction rule for \( \rightarrow \): we have to show that
  \( \{ \tau (\Gamma \vdash \varphi \vdash \psi) \} \vdash_{BCK} \tau (\Gamma \vdash \varphi \rightarrow \psi) \) \((H \rightarrow)\).

  And this is true from the definition of \( \tau \). □
Lemma 5. Let $\tau$ be the translation of $\mathcal{L}$-sequents in $\mathcal{L}$-formulas defined in Theorem 3 and let $\Sigma_i$, $\Sigma$ be finite sequences of $\mathcal{L}$-formulas and $\sigma_i$, $\sigma$ $\mathcal{L}$-formulas, then
\[
\{\Sigma_i \vdash \sigma_i : i \in I\} \vdash_{L_{BCK}} \Sigma \vdash \Rightarrow \{\tau(\Sigma_i \vdash \sigma_i) : i \in I\} \vdash_{BCK} \tau(\Sigma \vdash \sigma)
\]

Proof. We use induction on the length of the proof.

- $n = 1$
  i. If $\Sigma \vdash \sigma$ is any of $\Sigma_i \vdash \sigma_i$, then it is evident.
  ii. If $\Sigma \vdash \sigma$ is the axiom, then we have to prove that $\emptyset \vdash_{BCK} \tau(\varphi \vdash \varphi)$, that is, $\emptyset \vdash_{BCK} \varphi \rightarrow \varphi$, and this is true in $BCK$.

- $n > 1$
If $\Sigma \vdash \sigma$ is obtained by applying a rule $\{\Delta_j : j \leq 2\}$ of $L_{BCK}$, then using induction hypothesis we have
\[
\{\tau(\Sigma_i \vdash \sigma_i) : i \in I\} \vdash_{BCK} \tau(\Delta_j : j \leq 2) \text{ for all } j \leq 2
\]
and applying Lemma 4 we obtain
\[
\{\tau(\Sigma_i \vdash \sigma_i) : i \in I\} \vdash_{BCK} \tau(\Sigma \vdash \sigma).
\]

Now we are ready to prove the equality mentioned above.

Theorem 6. $\vdash_{S_{BCK}} \Rightarrow \vdash_{BCK}$

Proof.

$\supseteq$): Suppose $\Gamma \vdash_{BCK} \xi$. We will prove that $\Gamma \vdash_{S_{BCK}} \xi$ by induction on the length of the proof.

- $n = 1$
  i. If $\xi \in \Gamma$ then it is trivial.
  ii. If $\xi$ is an instance of three axioms called $B$, $C$ and $K$, then by [7, corollary 2.8.] we obtain that $\emptyset \vdash_{L_{BCK}} \emptyset \vdash \xi$, and applying the definition of $S_{BCK}$ we obtain $\emptyset \vdash_{S_{BCK}} \xi$, so $\Gamma \vdash_{S_{BCK}} \xi$.

- $n > 1$
If $\xi$ is obtained by modus ponens from the previous formulas $\psi$ and $\psi \rightarrow \xi$, we have by the induction hypothesis $\Gamma \vdash_{S_{BCK}} \psi$ and $\Gamma \vdash_{S_{BCK}} \psi \rightarrow \xi$. Finally, from $\{\psi, \psi \rightarrow \xi\} \vdash_{S_{BCK}} \xi$, we obtain $\Gamma \vdash_{S_{BCK}} \xi$.

$\subseteq$): We suppose that $\Gamma \vdash_{S_{BCK}} \xi$, that is, $\{\emptyset \vdash \gamma : \gamma \in \Gamma\} \vdash_{L_{BCK}} \emptyset \vdash \xi$, then using Lemma 5 we have $\{\tau(\emptyset \vdash \gamma) : \gamma \in \Gamma\} \vdash_{BCK} \tau(\emptyset \vdash \xi)$, so $\Gamma \vdash_{BCK} \xi$. $\square$

Now we are ready to prove the main result of this paper.
Theorem 7. $\mathcal{G}_{BCK}$ and $BCK$ are equivalent, that is, the translations $\tau$ from $\mathcal{G}_{BCK}$ to $BCK$ and $\rho$ from $BCK$ to $\mathcal{G}_{BCK}$ defined in Theorem 3 satisfy the following properties:

(i) $\{\Gamma \vdash \gamma : i \in I\} \vdash_{LBCK} \Gamma \vdash \gamma \iff \{\tau(\Gamma \vdash \gamma_i) : i \in I\} \vdash_{BCK} \tau(\Gamma \vdash \gamma)$.

(ii) $\phi \vdash \tau(\rho(\phi))$.

Proof. It is straightforward to check these properties from Theorems 3 and 6. 

Corollary 8. $\mathcal{G}_{BCK}$ is algebraizable with equivalent quasivariety semantics $BCK$, with the translations $\tau$ from $\mathcal{G}_{BCK}$ in $BCK$ and $\rho$ from $BCK$ in $\mathcal{G}_{BCK}$ defined by:

$\tau_{(m,1)}(p_0, \ldots, p_{m-1} \vdash q_0) = \begin{cases} p_0 \to (\ldots \to (p_{m-1} \to q_0) \ldots) \approx q_0 \to q_0 & \text{if } m \neq 0 \\
q_0 \approx q_0 \to q_0 & \text{if } m = 0 \end{cases}$

$\rho(p_0 \approx p_1) = \{p_0 \vdash p_1, p_1 \vdash p_0\}$.

Proof. It follows from the theorem and the fact that the quasivariety $BCK$ is an equivalent algebraic semantics of $BCK$, with defining equation $X = X \to X$ and equivalence formulas $\{X \to Y, Y \to X\}$ (see [3, theorem 5.11]). 

Corollary 9. $\mathcal{G}_{BCK}$ does not have the deduction-detachment theorem (DDT).

Proof. Suppose that $\mathcal{G}_{BCK}$ has the DDT. Then, as $\mathcal{G}_{BCK}$ is equivalent to $BCK$, it is easy to see that $BCK$ has the DDT, contradicting the already known fact [4, example 7.3] that $BCK$ does not have the DDT. 

Corollary 10. $\mathcal{G}_{BCK}$ has the following local deduction-detachment theorem (LDDT)

$T; (\Gamma \vdash \varphi) \vdash_{LBCK} \Pi \vdash \xi \iff T \vdash_{LBCK} E_i(\tau(\Gamma \vdash \varphi) \cdot \tau(\Pi \vdash \xi))$ for some $i < \omega$,

where $E_i(\tau(\Gamma \vdash \varphi) \cdot \tau(\Pi \vdash \xi)) = \tau(\Gamma \vdash \varphi) \cdot \ldots \cdot \tau(\Gamma \vdash \varphi) \vdash \tau(\Pi \vdash \xi)$.

Proof. It is easy to see from the theorem and the fact that $BCK$ has the LDDT [6, example 2.1], that $\mathcal{G}_{BCK}$ has this LDDT. 

78
References


