THE CLASS OF PRECOMPLETE ŁUKASIEWICZ’S MANY-VALUED LOGICS AND THE LAW OF PRIME NUMBERS GENERATION

Abstract
A propositional formula (a function) whose iteration generates classes of prime numbers has been found. Some properties of these classes are discussed.

In 1922 J. Łukasiewicz introduced the notion of many-valued logical matrix (see [5]). For my purposes here, by Łukasiewicz’s logical matrix $M_{n+1}$ I will mean the following one:

$$M_{n+1} = \langle V, \sim, \to, \{n\} \rangle \quad (n \geq 2, n \in N),$$

where $V = \{0, 1, 2, \ldots, n\}$, $\{n\}$ is the set of designated values.

Functions $\sim x$ (negation) and $x \to y$ (implication) are defined on the set $V$ in the following way:

$$\sim x = n - x,$$

$$x \to y = \min(n, n - x + y).$$

Functions $x \lor y$ (disjunction) and $x \land y$ (conjunction) are defined as follows:

$$x \lor y = (x \to y) \to y = \max(x, y),$$

$$x \land y = \sim (\sim x \lor \sim y) = \min(x, y).$$

Let $L_{n+1}$ denote the set of all superpositions of $\sim x$ and $x \to y$. In turn the set of all $n+1$-valued functions will be denoted by $P_{n+1}$; $P_{n+1}$ is the set of all $n+1$-valued functions, which correspond to Post’s $n+1$-valued logics [6]. Thus, matrices of Post’s $n+1$-valued logics are functionally complete systems of functions. A system of functions $\mathcal{R} = \{f_1, f_2, \ldots, f_k, \ldots\}$ from $P_{n+1}$ is called functionally complete if every function from $P_{n+1}$ is defined by a superposition of functions from $\mathcal{R}$. The set of functions $\mathcal{R}$ is called functionally precomplete in $P_{n+1}$ if every enlargement $\{\mathcal{R}, f\}$
(= \mathcal{R} \cup \{f\}) of the set \mathcal{R} by a function f such that f \in P_{n+1} and f \not\in \mathcal{R}
is functionally complete.

**Theorem 1.** The set of functions \( L_{n+1} \) is functionally 
precomplete in \( P_{n+1} \) iff \( n \) is a prime number \([1]\).

Thus, we have a new definition of a prime number using a notion of 
functional precompleteness. Moreover, now we can define a prime number 
in terms of classes of valid formulae. Let us define a matrix \( M_{n+1} \), which 
differs from matrix \( M_{n+1} \) only by the definition of implication \( x \rightarrow^K y \):

\[ M_{n+1}^K = \langle V, \sim, \rightarrow^K, \{n\} \rangle \quad (n \geq 3, n \in N), \]

where

\[
x \rightarrow^K y = \begin{cases} 
  x, & \text{if } 0 < x < n, \ (x, y) \neq 1 \text{ and } x + y \leq n, \ (i) \\
  y, & \text{if } 0 < x < n, \ (x, y) \neq 1 \text{ and } x + y > n, \ (ii) \\
  y, & \text{if } 0 < x = y < n, \ (iii) \\
  x \rightarrow y \text{ otherwise,} \ (iv) 
\end{cases}
\]

where \((x, y) \neq 1\) denotes that \( x \) and \( y \) are not relatively prime numbers, i.e. \( x \) and \( y \) have a common divisor which is different from 1.

Let \( K_{n+1} \) denote the set of all superpositions of \( \sim x \) and \( x \rightarrow^K y \).

**Lemma 1.** For any \( n \geq 3, n \) is a prime number iff \( n \in K_{n+1} \).

Thus, Lemma 1 gives us a new definition of a prime number. If one introduces, in the usual way, a propositional language and an evaluation function with the range \( V = \{0, 1, 2, \ldots, n\} \), this results in the fact that every prime number is defined by the corresponding class of valid formulae.

It is in connection with this point that there arises the question about 
functional properties of the set \( K_{n+1} \).

**Lemma 2.** For any \( n \geq 3 \) such that \( n \) is a prime number, \( K_{n+1} = L_{n+1} \).

**Proof.**

1. \( L_{n+1} \subseteq K_{n+1} \).
2. \( x \rightarrow^1 y = \sim y \rightarrow^K \sim x \).

\(^1\)This result was rediscovered twice (see [2] and [7]).
(B) \( x \rightarrow^S y = x \rightarrow^1 ((y \rightarrow^1 y) \rightarrow^1 y) \).
(C) \( x \rightarrow^2 y = \sim y \rightarrow^S \sim x \).
(D) \( x \rightarrow^3 y = \sim ((y \rightarrow^K x) \rightarrow^K (y \rightarrow^K x)) \rightarrow^K (x \rightarrow^K y) \).
(E) \( x \lor^1 y = (x \rightarrow^3 y) \rightarrow^3 y \).
(F) \( x \rightarrow^4 y = ((x \rightarrow^K y) \rightarrow^2 (\sim y \rightarrow^K \sim x)) \lor^1 ((\sim y \rightarrow^K \sim x) \rightarrow^2 (x \rightarrow^K y)) = x \rightarrow y = \min(n, n - x + y) \).

II. \( K_{n+1} \subseteq L_{n+1} \).

From the definition of \( x \rightarrow^K y \) it follows that the set \( K_{n+1} \) is not functionally complete for any \( n \geq 2 \). At least, the functions \( \sim x \) and \( x \rightarrow^K y \) preserve the set of values \( \{0, n\} \). But, as we have shown above, \( L_{n+1} \) is included in \( K_{n+1} \). Since the set \( L_{n+1} \) is functionally precomplete when \( n \) is a prime number (Theorem 1), then \( K_{n+1} \subseteq L_{n+1} \).

Thus, \( K_{n+1} = L_{n+1} \). \( \square \)

From Lemma 1, Lemma 2 and the properties of \( L_{n+1} \) it follows that

**Theorem 2.** For any \( n \geq 3 \), \( n \) is a prime number iff \( K_{n+1} = L_{n+1} \).\(^2\)

**Note.** The function \( x \rightarrow^S y \) (see the formula (B)) is a Sheffer’s stroke for \( K_{n+1} \). Let \( S_{n+1} \) denote the set of all superpositions of the function \( x \rightarrow^S y \). Then for any \( n \geq 3 \), \( n \) is a prime number iff \( S_{n+1} = K_{n+1} \) [4].

The above definition of \( x \rightarrow^K y \) imposes rather strong restrictions for the case when \( x < y \). Thus in the definition \( x \rightarrow^K y \) we can reject the clause (i) and the condition \( (x + y) > n \) in the clause (ii). This function we denote by \( x \rightarrow^K' y \):

\[
\begin{align*}
x \rightarrow^K' y &= \begin{cases} 
y, & \text{if } 0 < x < y < n \text{ and } (x, y) \neq 1, \quad (i) 
y, & \text{if } 0 < x = y < n, \quad (ii) 
x \rightarrow y, & \text{otherwise}, \quad (iii)
\end{cases}
\end{align*}
\]

Let \( K'_{n+1} \) denote the set of all superpositions of \( \sim x \) and \( x \rightarrow^K' y \).

**Theorem 3.** For any \( n \geq 3 \), \( n \) is a prime number iff \( K'_{n+1} = L_{n+1} \).\(^3\)

\(^2\)Note that Theorem 2 takes place also for the case \( n = 2 \). But then this case is subject to a separate treatment else we are to redefine the function \( x \rightarrow^K y \), which makes the proof more complicated.
The proof is similar to that of Theorem 2 but the proof of Lemma 2 (I) becomes somewhat more complicated:

\[(L) \quad x \rightarrow^1 y = \sim ((y \rightarrow^{K'} x) \rightarrow^{K'} (y \rightarrow^{K'} x)) \rightarrow^{K'} (x \rightarrow^{K'} y),\]

\[(M) \quad x \lor^1 y = (x \rightarrow^1 y) \rightarrow^1 y,\]

\[(N) \quad x \rightarrow^2 y = ((x \rightarrow^{K'} y) \rightarrow^{K'} (\sim y \rightarrow^{K'} x)) \lor^1 ((\sim y \rightarrow^{K'} x) \rightarrow^{K'} (x \rightarrow^{K'} y)),\]

\[(O) \quad x \lor^{K'} y = (x \rightarrow^{K'} y) \rightarrow^{K'} y,\]

\[(P) \quad x \lor y = (x \lor^{K'} y) \lor^1 (y \lor^{K'} x) = \max(x, y),\]

\[(Q) \quad x \rightarrow^3 y = (x \rightarrow^{K'} y) \lor (\sim y \rightarrow^{K'} x),\]

\[(R) \quad x \lor^3 y = (x \rightarrow^3 y) \rightarrow^3 y,\]

\[(S) \quad x \rightarrow^4 y = (((x \lor^3 y) \rightarrow^2 (y \lor x)) \rightarrow^1 (x \rightarrow^3 y),\]

\[(T) \quad x \rightarrow^5 y = ((x \rightarrow^4 y) \lor^1 (\sim y \rightarrow^4 x)) = x \rightarrow y = \min(n, n - x + y).\]

Theorem 2 and Theorem 3 together imply

**Theorem 4.** For any \( n \geq 3 \), \( n \) is a prime number iff \( K_{n+1} = K'_{n+1}. \)

The theorem suggests the following line of thought: let us substitute, in the proof of Lemma 2(I) the function \( x \rightarrow^{K'} y \) for \( x \rightarrow^{K} y \) and denote the resulting sequence of formulae by (A') –(F'). It is not difficult to show that the formula (F'):

\[ x \rightarrow^{4'} y = (((x \rightarrow^{K'} y) \rightarrow^{2'} (\sim y \rightarrow^{K'} x)) \lor^{1'} ((\sim y \rightarrow^{K'} x) \rightarrow^{2'} ((x \rightarrow^{K'} y))^{3} \]

defines \( \text{Łukasiewicz’s implication} \) \( x \rightarrow y \) only for the first five odd prime numbers: 3, 5, 7, 11 and 13. However, if \( n = 17, x = 2 \) and \( y = 12 \), then \( x \rightarrow^{4'} y = 15 \), while \( x \rightarrow y = 17 \). One can show that an iteration of formula (F') will induce the classes of prime numbers for which a new formula resulting from the iteration defines \( x \rightarrow y \). Suppose

\[ A_0 = (x \rightarrow^{K'} y) \rightarrow^{2'} (\sim y \rightarrow^{K'} x) \text{ and } \]

\[ B_0 = (\sim y \rightarrow^{K'} x) \rightarrow^{2'} (x \rightarrow^{K'} y). \]

Then

\[ C_0 = A_0 \lor^{1'} B_0. \]

\(^{3}\text{Let me note the difference between this formula and formula (N). Suppose } x = n \text{ and } y = 0. \text{ Then } x \rightarrow^2 y = n, \text{ while } x \rightarrow^{4'} y = 0 = x \rightarrow y.\)
Thus, the iteration consists in that one substitutes the implication \( \rightarrow \) for the disjunction \( \lor \) in the original formula \( C_0 \) (let us denote this operation by \([\rightarrow / \lor]\)), next applies the operation of conversion (\( REV \)) changing consequent with antecedent and finally links the formula after the operation \([\rightarrow / \lor]\) with that one after (\( REV \)) by disjunction \( \lor \). In the general case it has the following form:

\[
C_i = ([\rightarrow / \lor]C_{i-1}) \lor (REV([\rightarrow / \lor]C_{i-1})).
\]

Let \( P_i \) denote the class of prime numbers for which \( C_i = x \rightarrow y \). Then

\[
P_0 = \{3, 5, 7, 11, 13\},
\]

\[
P_1 = P_0 \cup \{17, 19\}.
\]

By means of the computer program developed by V. I. Shalak one can calculate other \( P_i \):

\[
P_2 = P_1 \cup \{23, 29, 31, 41, 43, 53, 59, 61\},
\]

\[
P_3 = P_2 \cup \{37, 47, 109\}.
\]

The class \( P_4 \) contains already 50 new prime numbers.\(^4\)

In fact the formula \( C_i \) is the law of prime numbers generation, more precisely, the law of classes of prime numbers generation. Let one note some unregularities of the classes \( P_i \). In what follows the prime number 223 is a member of the class \( P_8 \) while the class \( P_5 \) has already the prime number 757 as the greatest member.

**Theorem 5.** Every odd prime number is contained in some class \( P_i \).

**Hypothesis.** Every class \( P_i \) is finite.

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\(^4\)To simplify calculations, on the basis of the formula (P) one can substitute the function \( x \lor y = \max(x, y) \) in place of \( x \lor^1 y \) in \( C_i \).
References


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