Andrzej Indrzejczak

A NOTE CONCERNING BELIEF LOGIC

In the papers [3, pp. 157–190], [4], [5] professor Tokarz presents four propositional logics of belief operator. Each of them is characterized axiomatically and by algebraic semantics, then proofs of (weak) completeness are provided. However the proofs of soundness are omitted. It should be explained since one of them (LIB) is not in fact sound with respect to its semantics. The aim of this note is to correct this little error. In addition we make some rather obvious remarks concerning the relations of these logics with their alethic modal counterparts.

The basic Belief logic is LB which is axiomatically characterized by the following axioms and rules:

- **Ax.1** \(\alpha\) if \(\alpha \in \text{Taut}\)
- **Ax.2** \(B\alpha \leftrightarrow BB\alpha\)
- **Ax.3** \(\neg B\alpha \leftrightarrow B\neg B\alpha\)
- **Ax.4** \(B\neg \alpha \rightarrow \neg B\alpha\)
- **Ax.5** \(B(\alpha \rightarrow \beta) \rightarrow (B\alpha \rightarrow B\beta)\)
- **Mp** \(\vdash \alpha, \vdash \alpha \rightarrow \beta / \vdash \beta\)
- **RB** \(\vdash \alpha / \vdash B\alpha\)

Then we have two independent strengthenings:

- **LCB** = LB + Ax.6: \(B\alpha \vee B\neg \alpha\) (which is the same as changing Ax.4 or Ax.5 into equivalence) and
- **S5** = LB + Ax.7: \(B\alpha \rightarrow \alpha\)

The fourth logic LIB is the weakening of LB obtained by dropping Ax.4. But this weakening is only apparent. In what follows we show that Ax.4 is provable in LIB, so both sets of axioms are equivalent. Fortunately this error has no consequence for completeness theorem for LIB; Ax.4 has not been used in its proof.
1. \( \alpha \rightarrow (\neg \alpha \rightarrow \neg B \neg \alpha) \) by Ax.1
2. \( B \alpha \rightarrow B(\neg \alpha \rightarrow \neg B \neg \alpha) \) by 1, RB, Ax.5
3. \( B \alpha \rightarrow (B \neg \alpha \rightarrow B \neg B \neg \alpha) \) by 2, Ax.1, Ax.5
4. \( \neg B \neg \alpha \rightarrow B \neg B \neg \alpha \) by Ax.3
5. \( B \alpha \rightarrow B \neg B \neg \alpha \) by 3, 4, Ax.1
6. \( B \neg B \neg \alpha \rightarrow \neg B \neg \alpha \) by Ax.3
7. \( B \neg \alpha \rightarrow \neg B \alpha \) by 5, 6, Ax.1

The source of mistake is Ax.3 which is too strong for LIB as an equivalence. We can easily show that in half it is nonvalid in the class of FA (Filter Algebras) which is claimed to be adequate semantics for LIB.

Consider any Filter Algebra \( F = < A, *, F > \) such that \( A = < A, -, \cap, \cup > \) is a non-trivial Boolean algebra, \( F \) is the improper filter (i.e., \( F = A \)), and \( * \) is its characteristic function. Then clearly for any \( h \in Hom(For(LIB), F) \) and any \( \alpha \in For(LIB), h(B \alpha) = 1 \) which implies that \( h(B \neg B \alpha \rightarrow \neg B \alpha) = 0 \).

In order to obtain adequate axiomatics for LIB we must drop not only Ax.4 but also replace Ax.3 with its weaker version:

Ax.3’ \( \neg B \alpha \rightarrow B \neg B \alpha \)

This formula is sufficient for further strengthenings because its converse is derivable by Ax.4. In this context we can mention that Ax.2 is also partially redundant in LIB and LB, it is sufficient to have:

Ax.2’ \( B \alpha \rightarrow BB \alpha \)
since its converse is derivable by Ax.3’.

In effect we propose non-redundant axiomatics for LIB with Ax.2’, Ax.3’ in place of original ones, from which LB is achieved by adding Ax.4 (or alternatively the converse of Ax.3’). In the case of S5 we can do without Ax.2’ and Ax.4, which is well known, and in LCB Ax.2’ is redundant, which the following proof makes evident:

1. \( \neg B \neg \alpha \rightarrow B \alpha \) by Ax.6 and Ax.1
2. \( B \neg B \neg \alpha \rightarrow BB \alpha \) by 1, RB, Ax.5
3. \( \neg B \neg \alpha \rightarrow B \neg B \neg \alpha \) by Ax.3’
4. \( B \alpha \rightarrow \neg B \neg \alpha \) by Ax.4
5. \( B \alpha \rightarrow BB \alpha \) by 4, 3, 2, Ax.1

Thus revised axiomatics shows connections of LIB, LB, LCB with their alethic modal counterparts K45, K45D and K5DR, respectively.
The latter have simple characterization in terms of Kripke Frames (see for instance [1]). An exception is $K5DR$, which was rather not explored. Perzanowski [2, p. 311] investigated axiom \( R: M\alpha \rightarrow L\alpha \) in $KDR$; suitable relation of accessibility should have the property of Functionality:

For each world in the Frame there is one and only one accessible world.

To obtain Frames for $K5DR$ we should add the condition of Secondary Reflexivity:

If $w_2$ is accessible from $w_1$ then $w_2$ is accessible from itself, for any worlds $w_1$, $w_2$ (note that any relation which is functional and secondary reflexive is also trivially transitive and Euclidean).

Soundness of $K5DR$ with respect to this semantics is easy to establish. Proof of completeness goes along the lines of Henkin proof as for any other well known modal logic. We must only prove that relation of accessibility in the Canonical Model satisfies required properties.

Let $\Gamma$, $\Delta$, $\Theta$ be maximal sets. Proof that for each $\Gamma$ there is $\Delta$ such that \( \{ \varphi : L\varphi \in \Gamma \} \subseteq \Delta \) is as in KD by $L\varphi \rightarrow M\varphi$. To prove that it is unique assume that $\{ \varphi : L\varphi \in \Gamma \}$ is contained in $\Delta$ and in $\Theta$ and that $\Delta \neq \Theta$. Thus for some $\varphi, \xi \in \Delta$ and $\xi \notin \Theta$, hence $L\varphi \notin \Gamma$, so by maximality of $\Gamma, L\neg L\varphi \in \Gamma$ and by axiom $R$, $L\neg L\varphi \in \Gamma$. So $\neg \varphi \in \Delta$ which is impossible by consistency of $\Delta$, hence $\Delta = \Theta$.

In order to prove secondary reflexivity assume that $\{ \varphi : L\varphi \in \Gamma \} \subseteq \Delta$ and that there is some $\varphi$ such that $L\varphi \in \Delta$ but $\varphi \notin \Delta$. So $L\varphi \notin \Gamma$ and by maximality, $\neg L\varphi \in \Gamma$, then by axiom $3'$: $L\neg L\varphi \in \Gamma$ thus $\neg L\varphi \in \Delta$, which makes $\Delta$ inconsistent; so $\{ \varphi : L\varphi \in \Delta \} \subseteq \Delta$.

As a result of above considerations we have some more adequacy theorems:

$\text{LIB}$ is determined by the class of Kripke Frames with transitive and Euclidean relation of accessibility.

$\text{LB}$ is determined by the class of Kripke Frames with transitive, Euclidean and serial relation of accessibility.

$\text{LCB}$ is determined by the class of Kripke Frames with functional and secondary reflexive relation of accessibility.

$\text{K45}$ is determined by the class of Filter Algebras.

$\text{K45D}$ is determined by the class of Proper Filter Algebras.

$\text{K5DR}$ is determined by the class of Ultrafilter Algebras.
References


Department of Logic
University of Łódź
Matejki 34a
90–237 Łódź
Poland