A NOTE ON DERIVATION RULES IN MODAL LOGIC

The traditional Hilbert-style deductive apparatus for Modal logic in a broad sense (incl. temporal, dynamic, epistemic etc. logics) seems to have exhausted the potential of its standard form i.e. with only rules of inference being Modus Ponens, Necessitation and Substitution, plus Quantifier rules for the first-order versions. Various expressively strong and/or goal-oriented extended modal languages have been recently introduced which cannot be adequately axiomatized within that traditional framework and require specific additional rules of inference. Besides, introduction of additional rules has proved to be efficient for increasing the expressiveness of the language by means of eliminating unwanted models (those in which the new rules do not preserve validity) and thus depicting more precisely the intended semantics.

A notable example of a class of rules having recently become increasingly popular are the so-called in [10] “non-$\xi$ rules”:

\[ \frac{\neg^{\xi}(p_0, \ldots, p_n) \rightarrow \varphi \text{ for some } p_0, \ldots, p_n \text{ not occurring in } \varphi}{\varphi} \]

which in the presence of the Substitution rule is deductively equivalent to the infinitary but uniform version

\[ \frac{\neg^{\xi}(p_0, \ldots, p_n) \rightarrow \varphi \text{ for all } p_0, \ldots, p_n}{\varphi} \]

These rules generalize the idea of Gabbay’s irreflexivity rule (see [1]), an instance of which is obtained for $\xi = \Box p \rightarrow p$.

The rule $(N\xi R)$ added to the minimal normal logic $\mathcal{K}$ axiomatizes the class of frames in which the formula $\xi(p_0, \ldots, p_n)$ is refutable at every world by an appropriate valuation depending on that world. For example, the irreflexivity rule axiomatizes the class of irreflexive frames. A general completeness result goes along the lines of the classical canonical model
technique modified as follows: the canonical model is built only from those maximal theories which are closed under the rule \((N\xi R)_{mL}\) and thus is bound to be carried by a standard frame for the logic, since every such theory does not contain \(\xi(p_0, \ldots, p_n)\) for some \(p_0, \ldots, p_n\). Moreover, non-\(\xi\) rules can be used in combination and together with other axioms; for details and some general results see [10].

Unfortunately, the non-\(\xi\) rules in the form mentioned above work only in “temporalized” languages which together with each modality \(L\) contain (definable) its “mirror image” corresponding to the inverse of the relation semantically attached to \(L\). In the general case, however, (e.g. in monomodal or classical dynamic logic) the rule \((N\xi R)\) has to be replaced by a much stronger generalized rule scheme, typically:

\[
(N\xi R)^* \quad \frac{U(\neg\xi(p_0, \ldots, p_n) \rightarrow \varphi)}{U(\varphi)}
\]

where \(U(*)\) is a specific formula scheme called in [4] admissible form. Here is a general inductive definition of an admissible form in a modal language \(L\):

- \(*\) is an admissible form in \(L\);
- if \(U(*)\) is an admissible form of \(L\) and \(\psi\) is a formula of \(L\) then \(\psi \rightarrow U(*)\) is an admissible form in \(L\);
- if \(U(*)\) is an admissible form in \(L\) and \(L\) is a (box-)modality in \(L\) then \(U(U(*))\) is an admissible form in \(L\).

Every admissible form can be equivalently represented in a uniform shape:

\[
U(*) = \psi_0 \rightarrow L_1(\psi_1 \rightarrow \ldots \rightarrow L_n(\psi_n \rightarrow *)) \ldots)
\]

where \(L_1, \ldots, L_n\) are box-modalities in \(L\) and \(\psi_1, \ldots, \psi_n\) are formulae in \(L\), some of them possibly \(\top\). The number \(n\) is called a modal depth of \(U(*)\).

The admissible-form version of non-\(\xi\) rules or their relatives have been employed and studied in [1], [4], [6], [3], and, in disguise, in [9], [10] and [8].

As shown in [2], in temporalized languages the form \(U(*)\) can be turned inside out and thus the rule \((N\xi R)^*\) can be reduced to \((N\xi R)\).
This, however, is generally impossible in languages with uni-directional
modalities, as examples in [3] and [10] testify.

In this note we show that in an expressive enough modal language \( L \) and a strong enough (see further) logic in that language all admissible-
form-type rules collapse to their simplest versions. As a consequence, the
deductive systems introduced in [6], [3], [8] can be accordingly simplified.

Some definitions follow.

For the purpose of this note a modal language \( L \) will be called expressive enough if it contains (definable):

- a universal modality \( A \) satisfying the truth condition \( M \models A\varphi[t] \) iff \( M \models \varphi \). The dual \( \neg A \neg \) will be denoted by \( E \).
- a formula \( \sigma(p) \) which says “\( p \) is only true at the current world”, i.e. \( M \models \sigma(p)[t] \) iff \( p \) is true only at \( t \).

Let \( \Lambda \) be a normal logic in an expressive enough modal language \( L \).
We shall call \( \Lambda \) strong enough if:

1. All instances of the following schemata are theorems of S:
   (A1) the \( S5 \)-axioms for \( A \);
   (A2) \( A\varphi \rightarrow L\varphi \) for each box-modality \( L \) in \( L \);
   (A3) \( \sigma(p) \land \varphi \rightarrow A(\sigma(p) \rightarrow \varphi) \).

   and

2. The following rule is (derivable) in \( \Lambda \):

\[
\Sigma : \frac{\psi \rightarrow L(\sigma(p) \rightarrow \varphi) \text{ for some } p \text{ not occurring in } \varphi \text{ and } \psi}{\psi \rightarrow L\varphi}
\]

where \( L \) is any box-modality in \( L \).

Examples of recently studied strong enough modal logics are:

- the logic with names, see [3];
- the logic with difference modality, see [7], [8];
- the logic with reference pointers, see [5].
Theorem. Let $\Lambda$ be a strong enough logic in a language $\mathcal{L}$ which contains the (derivable) rule

$$(R_\chi): \frac{\chi \rightarrow \varphi}{\varphi}$$

for some fixed formula $\chi$. Then for every admissible form $U(*)$ the rule

$$(R_{U\chi}^U): \frac{U(\chi \rightarrow \varphi)}{U(\varphi)}$$

is derivable in $\Lambda$.

Proof. Induction on the modal depth $\partial(U)$ of the form $U$.

If $\partial(U) = 0$ then $U(*) = \psi \rightarrow *$ and

$\psi \rightarrow (\chi \rightarrow \varphi) \vdash \chi \rightarrow (\psi \rightarrow \varphi) \vdash \psi \rightarrow \varphi$

Now let $\partial(U) = n > 0$ and for all admissible forms with depth $\leq n - 1$ the statement holds.

We shall reduce the rule $(R_{U\chi}^U)$ to some rule $(R_{U'}^U)$ of depth $n-1$ using the following main lemma which enables turning an admissible form inside-out.

Lemma. For any modality $\Box$ in $\mathcal{L}$ and formulae $\alpha$ and $\beta$ not containing the variable $p$, each of the formulae $\alpha \rightarrow \Box \beta$ and $\sigma(p) \land E(\Diamond \sigma(p) \land \alpha) \rightarrow \beta$ is derivable from the other in $\Lambda$.

Proof. We shall sketch the derivations omitting some simple steps based on K-derivations for $\Box$ and S5-derivations for $A$. First we do the derivation

$\alpha \rightarrow \Box \beta \vdash \sigma(p) \land E(\Diamond \sigma(p) \land \alpha) \rightarrow \beta$.

Assuming that $\alpha \rightarrow \Box \beta$ is derived we subsequently get:

1. $\vdash \Diamond \neg \beta \rightarrow \neg \alpha$;
2. $\vdash A(\Diamond \neg \beta \rightarrow \neg \alpha)$
3. $\vdash \sigma(p) \land \neg \beta \rightarrow A(\sigma(p) \rightarrow \neg \beta)$, by (A3);
4. $\vdash \sigma(p) \land \neg \beta \rightarrow AA(\sigma(p) \rightarrow \neg \beta)$, by (A1) and 3;
5. $\vdash \sigma(p) \land \neg \beta \rightarrow A\Box(\sigma(p) \rightarrow \neg \beta)$, by (A2) and 4;
6. \( \vdash \sigma(p) \land \lnot \beta \rightarrow A(\diamond \sigma(p) \rightarrow \lnot \beta) \), by 5;
7. \( \vdash \sigma(p) \land \lnot \beta \rightarrow A(\diamond \sigma(p) \rightarrow \lnot \alpha) \), by 2 and 6;
8. \( \vdash \sigma(p) \land E(\diamond \sigma(p) \land \alpha) \rightarrow \beta \), by 7.

Now we show
\[
\sigma(p) \land E(\diamond \sigma(p) \land \alpha) \rightarrow \beta \vdash \alpha \rightarrow \Box \beta.
\]

Assume that \( \sigma(p) \land E(\diamond \sigma(p) \land \alpha) \rightarrow \beta \) is derived. Then:

1. \( \vdash E(\diamond \sigma(p) \land \alpha) \rightarrow (\sigma(p) \rightarrow \beta); \)
2. \( \vdash AE(\diamond \sigma(p) \land \alpha) \rightarrow A(\sigma(p) \rightarrow \beta); \)
3. \( \vdash \diamond \sigma(p) \land \alpha \rightarrow A(\sigma(p) \rightarrow \beta), \) by (A1) and 2;
4. \( \vdash \diamond \sigma(p) \land \lnot \beta \land \alpha \rightarrow A(\sigma(p) \rightarrow \beta), \) by 3;
5. \( \vdash \sigma(p) \land \lnot \beta \rightarrow A(\sigma(p) \rightarrow \gamma \beta), \) by (A3)
6. \( \vdash \diamond(\sigma(p) \land \lnot \beta) \rightarrow A(\sigma(p) \rightarrow \lnot \beta), \) by 5;
7. \( \vdash \diamond(\sigma(p) \land \lnot \beta) \rightarrow A(\sigma(p) \rightarrow \gamma \beta), \) by 6, (A2) and (A1);
8. \( \vdash \diamond(\sigma(p) \land \lnot \beta) \land \alpha \rightarrow A(\lnot \sigma(p)), \) by 4, 7;
9. \( \vdash \diamond(\sigma(p) \land \lnot \beta) \rightarrow E\sigma(p), \) by (A2);
10. \( \vdash \diamond(\sigma(p) \land \lnot \beta) \land \alpha \rightarrow \perp, \) by (8) and (9);
11. \( \vdash \alpha \rightarrow \Box(\sigma(p) \rightarrow \beta), \) by (10);
12. \( \vdash \alpha \rightarrow \Box \beta, \) by \( \Sigma; \)

The proof of the lemma is completed. \( \square \)

Now, let \( \mathcal{U}(\ast) = \psi_0 \rightarrow L_1(\psi_1 \rightarrow \ldots \rightarrow L_n(\psi_n \rightarrow \ast \ldots)) \) and \( \mathcal{U}^-(\ast) = \psi_1 \rightarrow L_2(\psi_2 \rightarrow \ldots \rightarrow L_n(\psi_n \rightarrow \ast \ldots)). \) Then the form
\[
\mathcal{V}(\ast) = \sigma(p) \land E(\diamond \sigma(p) \land \psi_0) \rightarrow \mathcal{U}^-(\ast)
\]
has a modal depth n-1 hence the rule \( (R^\chi_V) \) is derivable. Now it remains to apply the lemma:

\[
\mathcal{U}(\chi \rightarrow \varphi) \vdash_{\text{lemma}} \mathcal{V}(\chi \rightarrow \varphi) \vdash_{\text{ind.hyp.}} \mathcal{V}(\varphi) \vdash_{\text{lemma}} \mathcal{U}(\varphi).
\]

This completes the proof of the theorem. \( \square \)

By obvious modifications of the above proof, the theorem extends to rules with provisos like \( (N\xi R) \) and \( (N\xi R)_{inf}. \)

We end this note with two applications of the above Theorem.
Corollary 1. The rule \( \text{COV} \) in [3] can be reduced to
\[
\text{COV}_0 : \frac{c \rightarrow \varphi \text{ for some } p \text{ not occurring in } \varphi}{\varphi}
\]
and
\[
\text{COV}_1 : \frac{\psi \rightarrow \Box(c \rightarrow \varphi) \text{ for some } p \text{ not occurring in } \varphi \text{ and } \psi}{\psi \rightarrow \varphi}
\]

Corollary 2. The rule scheme \((\text{IR}_D^*)\) in the axiomatization of the modal logic with difference modality ([9], [8]) can be reduced to
\[
(\text{IR}_D)_0 : \frac{p \land \neg p \rightarrow \varphi \text{ for some } p \text{ not occurring in } \varphi}{\varphi}
\]
and
\[
(\text{IR}_D)_1 : \frac{\psi \rightarrow \Box(p \land \neg p \rightarrow \varphi) \text{ for some } p \text{ not occurring in } \varphi \text{ and } \psi}{\psi \rightarrow \Box \varphi}
\]

References


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