ON SOME OPERATORS ON PSEUDOVARIE TIES II

Dedicated to all my Friends

Abstract
This is a continuation of my paper On some operators on pseudovarieties [6]. We adopt the HSP Theorem of G. Birkhoff [1] and the Theorem of H. Lakser, R. Padmanabhan, C.R. Platt [11], for pseudovarieties. Some of the results were presented at the 48 Workshop on General Algebra, in Linz (Austria), 2–5 June, 1994.

An extended version of the paper will be published separately.

1. Preliminaries

Several varieties of algebras are defined by special identities for example normal or regular of another variety. Thus, the question of representing of the algebras and the lattice $L(P(V))$ of all subvarieties of $P(V)$ by the lattice $L(V)$ is important, where $P$ is an operator on varieties. Some interesting properties of the operators $N$ and $R$ are very useful in these considerations. In E. Graczyńska [5], [6], we have pointed some results in that direction. Here we present a generalization of these results for pseudovarieties of algebras, i.e. classes of finite algebras of a given type $\tau$ which are closed under the operators of forming subalgebras, homomorphic images and finite products of algebras (see S. Eilenberg [3], [4]):

**Definition 1.1.** [Eilenberg, Schützenberger] Given a sequence $\Sigma = (e_n : n \in N)$ of identities of a given type $\tau$. We say that a class $V$ of algebras of type $\tau$ is ultimately defined by $\Sigma$ if and only if any algebra $A$ of $V$ satisfies all but finite number of identities of $V$.

**Theorem 1.1.** [Eilenberg, Schützenberger [4], cf. [6]] Every nonempty pseudovariety of algebras of a given finite type is ultimately defined by a sequence of identities of type $\tau$. 

80
Following G. Gratzer [7], p. 152, we say that $X$ is an operator if for every class $K$ of finite algebras, $X(K)$ is also a class of finite algebras. If $X$ and $Y$ are operators, so is $XY$ defined by: $XY(K) = X(Y(K))$. $X^2$ will stand for $XX$. In the sequel, $H$, $S$, $P_{\text{fin}}$ shall denote the following operators on pseudovarieties (cf. [12]): $H$ – taking a homomorphic image of algebras, $S$ – taking subalgebras and $P_{\text{fin}}$ – for taking finite products of algebras (i.e. direct products of nonvoid finite families of algebras).

**Lemma 1.1.** If $X = H, S, P_{\text{fin}}$, then $X^2 = X$.

**Proof.** Similar as the proof of Lemma 1 of G. Gratzer [7], p. 152. □

**Theorem 1.2.** Let $K$ be a class of finite algebras. Then:

(i) $SH(K) \subseteq HS(K)$;

(ii) $P_{\text{fin}} H(K) \subseteq HP_{\text{fin}}(K)$;

(iii) $P_{\text{fin}} S(K) \subseteq SP_{\text{fin}}(K)$.

**Proof.** The proof of (i) follows from Lemma 3 (ii) p. 63, of (ii): from Lemma 3, p. 127 and Theorem 1, p. 152 of G. Gratzer [7], where direct products should be taken for finite families of algebras. The proof of (iii) is obvious. □

**Theorem 1.3.** Given a class $K$ of finite algebras of a given type $\tau$.

Then $HSP_{\text{fin}}(K)$ is the smallest pseudovariety of algebras, containing $K$.

**Proof.** The proof is similar as those of G. Gratzer [7], p. 153, via Lemma 1.1 and inclusions from Theorem 1.2. In that case, we say that $K$ generates $V$.

Let us note, that the following famous theorems remain valid for pseudovarieties (cf. [7] p. 124, 244). Here, as usually: $I$ is the operator of taking isomorphic copies of algebras, $(P_{\text{fin}})_S$ denotes the operator of the formation subdirect products of nonvoid, finite families of algebras, $(P_{\text{fin}})_P$ is the operator of prime finite products:

**Theorem 1.4.** [Birkhoff] Given a pseudovariety $V$ of algebras. Then every algebra of $V$ is a subdirect product of (finitely many) subdirectly irreducible algebras of $V$.

**Theorem 1.5.** [Jónsson] Let $V$ be a pseudovariety of algebras generated by a class $K$ of finite algebras. Let us assume, that the congruence lattice
of every algebra of $V$ is distributive. Then:

$$V = I(P_{fin})SHS(P_{fin})P(K).$$

2. Regular pseudovarieties

In E. Graczyńska [6] we investigated the operator $R$ on pseudovarieties, in connection with regular identities. The notion of regular identity was defined by J. Plonka in [13] (cf. B. Jónsson, E. Nelson [9], W. Taylor [16]). From now on, we consider only algebras of a given, finite type $\tau$, without constants.

Definition 2.1. [Plonka] An identity $p = q$ is regular if $\text{Var}(p) = \text{Var}(q)$. A variety $V$ is regular if all the identities satisfied in $V$ are regular.


The following example of a sum of a semilattice ordered system of trivial algebras was considered by B. Jónsson and E. Nelson in [9]: for a given type $\tau$, the one-element algebra $1_{\tau} = (\{1\}, (f_t : t \in T))$ belongs to every variety $V$ of type $\tau$. Denote by $2_{\tau} = (\{0, 1\}, (f_t : t \in T))$ the algebra defined by:

$$f_t(a) = \begin{cases} 
1 & \text{if } a_k = 1 \text{ for all } k \in \tau(t) \\
0 & \text{if } a_k = 0 \text{ for some } k \in \tau(t)
\end{cases}; \text{ for all } a \in \{0, 1\}^{\tau(t)}. \quad (1)$$

This definition is very useful, because of the following:

Lemma 2.1. [Jónsson, Nelson] An identity $p = q$ holds in $2_{\tau}$ iff it is regular.

Corollary 2.1. [Jónsson, Nelson] A variety $V$ is regular iff $2_{\tau} \in V$.

The results above were formulated in the form of Birkhoff’s type theorem (see G. Birkhoff [1], G. Gratzer [7], p. 152) by W. Taylor [16], p.4; where the algebra $2_{\tau}$ was considered under the name of sup-algebra of type $\tau$:  

82
Theorem 2.1. [Taylor] A variety $V$ is definable by regular identities iff it is closed under formation of products, subalgebras, homomorphic images and sup-algebras.

An analogous idea was invented by R. John [8]. Recall, that by the one-point extension of an algebra $A = (A, (F_t : t \in T))$ we mean the algebra $A^* = (A^*, (F^*_t : t \in T))$, where $A^* = A \cup \{0\}$ (a disjoint union) and:

$$F^*_t(a_1, ..., a_{n_t(t)}) = \begin{cases} F_t(a_1, ..., a_{n_t(t)}) & \text{if } a_1, ..., a_{n_t(t)} \in A; \\ 0 & \text{otherwise.} \end{cases} \quad (2)$$

Note, that the algebra $2_{\tau}$ is the one-point extension of the trivial algebra of type $\tau$.

Recall from [6] the generalization of the notion of regular variety for pseudovarieties:

Definition 2.2. A pseudovariety $V$ is regular if there exists a sequence $\Sigma$ of regular identities ultimately defining $V$. Otherwise, it is called non-regular.

Lemma 2.2. A pseudovariety $V$ is regular if and only if it is closed under the formation of sup-algebras.

Proof. Given a regular pseudovariety $V$ of finite algebras of a type $\tau$. Then $V$ is ultimately defined by a sequence $\Sigma$ of regular identities of type $\tau$. Therefore the algebra $2_{\tau}$ belongs to $V$. And vice versa: given a pseudovariety $V$ containing the algebra $2_{\tau}$. Let us assume, that the variety $V$ is ultimately defined by a sequence $\Sigma$ of identities of type $\tau$. Because the algebra $2_{\tau}$ satisfies all, but a finite number of identities of $\Sigma$, therefore we conclude that all (but a finite number) of identities of $\Sigma$ are regular. Therefore $V$ is ultimately defined by a sequence of regular identities, namely the sequence $\Sigma$ without the first (finite) number of nonregular identities.

According to S. Eilenberg (cf. J. E. Pin [12]), the family $L(\tau)$ of all pseudovarieties of type $\tau$ forms a complete lattice $L(\tau) = (L(\tau), \cap, \lor)$. Recall, that by $L(\tau) = (L(\tau), \cap, \lor)$ we denote the lattice of all varieties of type $\tau$. Obviously, for a given variety $V$ of algebras from $L(\tau)$, the class $\langle V \rangle_{fin}$, i.e. the class of all finite models of $V$ forms a pseudovariety of algebras (cf. [10], p. 222 and [6], §4).
Theorem 2.2. The algebra $\mathbf{2}_\tau$ generates the smallest regular pseudovariety of a given type $\tau$, namely $\text{HSP}_{\text{fin}}(\{\mathbf{2}_\tau\})$. Moreover:

\[ \text{HSP}_{\text{fin}}(\{\mathbf{2}_\tau\}) = (\text{HSP}(\{\mathbf{2}_\tau\}))_{\text{fin}}. \]

Proof. The first part of the theorem follows from the definition of a regular pseudovariety. The second part is a consequence of the Corollary of [10], p. 222.

Remark 2.1. Consider a finite type $\tau$. Then every set of identities of type $\tau$ can be arranged in a sequence $\Sigma = (e_n : n \in N)$. Consider the set $\Sigma^2$ and arrange it into the sequence – by the well known natural diagonal method. Then the sequence $\Sigma^2$ ultimately defines the pseudovariety $(\mathbf{V})_{\text{fin}}$.

By the same diagonal method, it can be shown that the sequence $(R(\tau))^2$ of all regular identities of a given finite type $\tau$ ultimately defines the smallest regular pseudovariety $\text{HSP}_{\text{fin}}(\{(\mathbf{2}_\tau)\})$.

In E. Graczyńska [6] we have noticed, that the operator $R$ is a homomorphism from the lattice $\mathcal{L}(\tau)$ into the lattice $\mathcal{L}(\tau)$.

Therefore it is natural, to ask the question about the operator $R$ on the lattice of pseudovarieties of type $\tau$. For a given pseudovariety $V$, $\mathcal{L}(V)$ denotes the lattice $(\mathcal{L}(V), \cap, \lor)$ of all pseudovarieties contained in $V$.

Theorem 2.3. Regular pseudovarieties of $\mathcal{L}(\tau)$ form a complete sublattice of the lattice $\mathcal{L}(\tau)$.

Proof. Given two regular pseudovarieties $V_1$ and $V_2$ of type $\tau$. Then the algebra $\mathbf{2}_\tau$ belongs to $V_1$ and $V_2$. Thus it belongs to the intersection $V_1 \cap V_2$ of pseudovarieties $V_1$ and $V_2$ in the lattice $\mathcal{L}(\tau) = (\mathcal{L}(\tau), \cap, \lor)$. Moreover, in that case, the algebra $\mathbf{2}_\tau$ belongs to any pseudovariety $W$ containing varieties $V_1$ and $V_2$, therefore the join $V_1 \lor V_2 = \cap\{W : V_1, V_2 \subseteq W, W \in \mathcal{L}(\tau)\}$ in the lattice $\mathcal{L}(\tau)$ is regular. Similarly, for infinite joins and meets of regular pseudovarieties.

Definition 2.3. (cf. [6]) For a given pseudovariety $V$ from $\mathcal{L}(\tau)$, $R(V)$ is the smallest regular pseudovariety from $\mathcal{L}(\tau)$ containing $V$.

Theorem 2.4. Given a nonregular pseudovariety $V$ of type $\tau$.

Then the pseudovariety $R(V)$ covers $V$ in the lattice $\mathcal{L}(\tau)$.

Proof. Given a pseudovariety $W$ of type $\tau$, with $V \subseteq W \subseteq R(V)$. Then obviously: $R(V) \subseteq R(W) \subseteq R(V)$, i.e. $R(V) = R(W)$. If $W$ is regular,
then $W = R(W) = R(V)$. Otherwise, i.e. if $W$ is not regular, then by
Definition 2.2 every algebra $A$ from $W$ satisfies a nonregular identity of
$V$. Assume that the sequence $\Sigma$ ultimately defines $W$. If $A \in W$, then
$A \in \text{Mod}(\Sigma - \Sigma_f)$, for some finite subsequence $\Sigma_f$ of $\Sigma$. Therefore every
algebra $A$ of $W$ belongs to $V$, by Lemma 2 of [2]. This ends the proof. \qed

**Definition 2.4.** A pseudovariety $V$ of $L(\tau)$ is called strongly nonregular,
if for every sequence $\Sigma$ of identities defining $V$ there exists an infinite
subsequence of $\Sigma$ consisting of strongly nonregular identities of type $\tau$,
i.e. being of the form $x = p(x, y)$, where $x, y$ are different variables and
$y \in \text{Var}(p(x, y))$.

2 denotes the two-element lattice $(\{0, 1\}, \land, \lor)$ with $0 < 1$.

**Theorem 2.5.** Given a strongly nonregular pseudovariety $V$. Then the op-
erator $R$ is a homomorphism from the lattice $\mathcal{L}(V)$ into the lattice
$\mathcal{L}(R(V))$ of all pseudovarieties contained in $R(V)$.

Similarly for a nonregular $V$ of a unary type.

**Proof.** Given two pseudovarieties $V_i \subseteq V$, $i \in \{1, 2\}$, where $V_i$ is ul-
timately defined by a sequence $\Sigma_i$ and $V$ is ultimately defined by a sequence
$\Sigma$, where $\Sigma$ is a subsequence of $\Sigma_i$ for $i \in \{1, 2\}$, which is possible, because of the assumption that $V_1, V_2 \subseteq V$, where $V$ is ultimately defined by a sequence $\Sigma$ (cf. Proposition 2 (iv) of [6]).
The following inclusion is obvious:
(a) $R(V_1 \cap V_2) \subseteq R(V_1) \cap R(V_2)$.
To prove the opposite inclusion, assume that an algebra $A \in R(V_1) \cap R(V_2)$.
Therefore the algebra $A$ satisfies the following sets of identities:
$E(\Sigma_i - \Sigma_{f_i}) \cap R(\tau)$, for some finite sets $\Sigma_{f_i} \subseteq \Sigma$, $i \in \{1, 2\}$.
Therefore we conclude, that $A \in R(W_1) \cap R(W_2)$ where $W_i$ is the variety of
algebras of a given type, defined by the set $\Sigma_i - \Sigma_{f_i}$, for $i \in 1, 2$. From the
assumption, that there is a nonregular identity, satisfied simultaneously in
$W_1$ and $W_2$, we conclude (similarly as in the proof of Theorem 2 of [2]) that
$A \in R(W_1 \cap W_2)$, i.e.: $A$ satisfies the set $E(\Sigma_i \cup \Sigma_{f_i}) \cap (\Sigma_{f_i} \cup \Sigma_{f_2}) \cap R(\tau)$
and therefore $A \in R(V_1 \cap V_2)$, by Proposition 5 (ii) of [6] and we obtain:
(b) $R(V_1 \cap V_2) \supseteq R(V_1) \cap R(V_2)$.
Finally, from the inclusions:
(c) $V_1 \cup V_2 \subseteq R(V_1) \cup R(V_2) \subseteq R(V_1 \cup V_2)$
and Theorem 2.4 we obtain:

85
(d) \( R(V_1 \lor V_2) = R(V_1) \lor R(V_2) \).
Therefore, we get that \( R \) is a homomorphism. \( \square \)

**Theorem 2.6.** If \( V \) is a strongly nonregular pseudovariety, then the operator \( R \) is an embedding of the lattice \( \mathcal{L}(V) \) into the lattice \( \mathcal{L}(R(V)) \).

Similarly for a nonregular \( V \) of a unary type.

**Proof.** The proof is similar as those of Theorem 5 of [6]. \( \square \)

**Theorem 2.7.** Let \( V, K \) be pseudovarieties where \( V \) is strongly nonregular (or \( V \) is nonregular of a unary type) and \( K \subseteq R(V) \). Then there are only two possibilities: \( K \subseteq V \) or \( K \) is regular.

**Proof.** The proof is similar as those of Proposition 6 of [6]. \( \square \)


**Theorem 2.8.** Given a strongly nonregular pseudovariety \( V \) of \( \mathcal{L}(\tau) \).
Then the lattice \( \mathcal{L}(R(V)) \) is isomorphic to the direct product of the two-element lattice \( 2 \) and \( \mathcal{L}(V) \). Similarly for a nonregular \( V \) of a unary type \( \tau \).

**Proof.** Consider the mapping \( h : \mathcal{L}(V) \times 2 \rightarrow \mathcal{L}(R(V)) \) defined by the rule: \( h((W,0)) = W, h((W,1)) = R(W) \), for every \( W \in \mathcal{L}(V) \).

To show that \( h \) is a homomorphism, let \( W_1, W_2 \in \mathcal{L}(V) \), for \( i = 1, 2 \). Then:
\[
h((W_1,0) \cap (W_2,0)) = W_1 \cap W_2 = h((W_1,0)) \cap h((W_2,0)).
\]
Note, that \( W_1 \cap W_2 = W_1 \cap R(W_2), W_1 \cap R(W_2) \) are not regular (cf. Proposition 2 (iv) [6]) and
\[
W_1 \cap W_2 \leq W_1 \cap R(W_2) \leq R(W_1) \cap R(W_2),
\]
while the pseudovariety \( R(W_1) \cap R(W_2) \) is regular and covers \( W_1 \cap W_2 \) in the lattice \( \mathcal{L}(\tau) \), by Theorem 2.4.

Therefore
\[
h((W_1,0) \cap (W_2,1)) = W_1 \cap W_2 = W_1 \cap R(W_2) = h((W_1,0)) \cap h((W_2,1)).
\]
Moreover,
\[
h((W_1,1) \cap (W_2,1)) = R(W_1 \cap W_2) = R(W_1) \cap R(W_2) = h((W_1,1)) \cap h((W_2,1))
\]
by Theorem 2.5 and we deduce that \( h \) is a meet-homomorphism.

Similarly, one can show that \( h \) is a join-homomorphism.

86
To prove that $h$ is an isomorphism, let us notice that $h((W_1,0))$ is a nonregular pseudovariety and $h((W_2,1))$ is a regular pseudovariety, therefore they are not equal for any $W_1, W_2 \in \mathcal{L}(V)$. If $h((W_1,1)) = h((W_2,1))$, i.e. $R(W_1) = R(W_2)$ then $W_1 = W_2$, by Theorem 2.6, which proves that $h$ is $1-1$.

To prove that $h$ is onto the lattice $\mathcal{L}(R(V))$, let $K' \in \mathcal{L}(R(V))$, i.e. $K' \subseteq R(V)$. By Theorem 2.7, there are only two possibilities: $K' \subseteq V$ or $K'$ is regular. In the first case: $K' = h((K',0))$. In the second case we consider the pseudovariety $K = K' \cap V \subseteq V$ and then, by Theorem 2.5, we obtain $h((K,1)) = R(K) = R(K' \cap V) = R(K') \cap R(V) = K'$, which ends the proof of the fact that $h$ is an isomorphism from the lattice $\mathcal{L}(V) \times 2$ onto the lattice $\mathcal{L}(R(V))$. □

References


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