ON COMPLETENESS OF INTERMEDIATE PREDICATE LOGICS WITH RESPECT TO KRIPKE SEMANTICS

In spite of the existence of many examples of incomplete logics, it is an important problem to find intermediate predicate logics complete with respect to Kripke frame (or Kripke sheaf) semantics because they are closed under substitution. But, most of known completeness proofs of finitely axiomatizable logics are difficult to apply to other logics since they are highly dependent on the specific properties of given logics.

So, it is preferable to find a general methods of completeness proof. We give some results on this problem using canonical formulas of propositional logics [8, 2].

1. Completeness results for some logics with the constant domain axiom

The first result is a generalization of Ono’s one ([3]). He proved the completeness of some intermediate logics of the form \( J^* + D \) by the method of canonical models, where \( J^* \) is the least predicate extension of an intermediate propositional logic \( J \) and \( D \) is the constant domain axiom \( \forall x(p(x) \lor q) \supset (\forall x p(x) \lor q) \).

For example, Ono’s proof of the completeness of \( J^* + D \) for \( J = H + (p \supset q) \lor (q \supset p) \) depends on the following properties:

1. \( J \) is a subframe logic ([1, 8]) with the finite embedding property. ([1])
2. The axiom \( (p \supset q) \lor (q \supset p) \) has the property: If \( N' \) is a subframe of \( N \) and \( \theta' \) is the restriction of \( \theta \) to \( N' \), then

\[
(N', \theta') \models (p \supset q) \lor (q \supset p) \Rightarrow (N, \theta) \models (p \supset q) \lor (q \supset p).
\]

So, it is possible to generalize Ono’s result if every subframe logic has the first and second properties. Since every intermediate subframe logic has
the finite embedding property, we have only to obtain an axiomatization of subframe logics with the property 2. Since Zakharyaschev’s canonical formula ([8]) \( X(M) \) does not have this property, we defined another canonical formula \( X'(M) \), which is equivalent to \( X(M) \) as axiom, and proved the following:

**Theorem 1.** [6] Let \( M \) be a finite Kripke frame and \((N, \theta)\) a Kripke model, where \( \theta \) is a valuation function.

1. \( N \not\models X'(M) \) iff there is a subreduction from \( N \) to \( M \).
2. If \( N' \) is a subframe of \( N \) and \( \theta' \) is the restriction of \( \theta \) to \( N' \), then

\[
(N', \theta') \not\models X'(M) \Rightarrow (N, \theta) \not\models X'(M).
\]

So, we have the following completeness result.

**Theorem 2.** [6] Let \( J \) be a subframe logic. Then \( J_* + D \) and \( J_* + D + K \) are strongly Kripke complete.

By the method of canonical models, the following can be obtained using Jankov’s characteristic formula:

**Theorem 3.** [6] Let \( J \) be a tabular logic. Then \( J_* + D \) is strongly Kripke complete.

An analogue of Theorem 2 holds for some cofinal subframe logics ([8]).

**Theorem 4.** [6] If a cofinal subframe logic \( J \) contains \( N_7 (= \neg \neg p \supset p) \), then \( J_* + D + K \) is strongly Kripke complete.

However these results cannot be extended to arbitrary cofinal subframe logic \( J \) because Skvortsov’s proved the incompleteness of \( J_* + D + K \) for a cofinal subframe logic \( J \).

2. The existence of complete but not strongly complete intermediate propositional logics.

Firstly we give the definition of strong completeness.

A **theory** is a pair \((\Gamma, \Delta)\) of sets of formulas. A theory \((\Gamma, \Delta)\) is said to be **L-consistent** for a logic \( L \) if
for every \( \gamma_1, \ldots, \gamma_m \in \Gamma \) and \( \delta_1, \ldots, \delta_n \in \Delta \). A logic \( L \) is said to be strongly Kripke complete if, for every \( L \)-consistent theory \((\Gamma, \Delta)\), there is a model \((M, \theta)\) of \((\Gamma, \Delta)\) such that every axiom of \( L \) is valid in \( M \). Hence a strongly complete logic is complete by definition.

One of the difficulty to obtain a completeness result is the existence of complete but not strongly complete intermediate propositional logics (the first example was essentially stated in Shehtman’s incompleteness result [5]) since known completeness proofs for intermediate predicate logics, in fact, prove their strong completeness. We give a necessary condition for strong completeness by means of the category of Kripke frames and p-morphisms, and show that there are quite many complete but not strongly complete propositional logics.

**Definition 5.** Let \( S = (\{M_n\}, \{f_{mn}\}) \) be an inverse system of finite Kripke frames and onto p-morphisms.

The pair of a Kripke frame \( M \) and a set of onto p-morphisms \{\( f_n \)\} from \( M \) to \( M_n \) is said to be a cone for \( S \) if \( f_n = f_{mn} \circ f_m \) holds for every \( m, n \).

**Theorem 6.** If an intermediate propositional logic \( J \) is strongly complete, then for any inverse system \( S = (\{M_n\}, \{f_{mn}\}) \) of finite Kripke frames in which \( J \) is valid and onto p-morphisms, there exists a cone \((M, f_n)\) for \( S \) such that \( J \) is valid in \( M \).

**Remark 7.** If \( J \) is locally tabular, then the converse holds.

**Proof (sketch).** We associate a set of distinct propositional variables \( \{p^n_u : u \in M_n\} \) to each \( M_n \). Define a theory \((\Gamma_1 \cup \Gamma_2, \Delta)\) as follows:

\[
\begin{align*}
\Gamma_1 &= \{Y(M_n) : n \in \omega\}, \\
\Gamma_2 &= \{p^n_u = \bigcup_{f_{mn}(v) \geq u} p^m_v : u \in M_n, \ m \geq n\}, \\
\Delta &= \{p(M_n \setminus \{0_{M_n}\}) : n \in \omega\},
\end{align*}
\]

where \( Y(M) \) denotes the assumption part of Jankov’s characteristic formula of \( M \).

Then we can prove the following claims.
Claim. \((\Gamma_1 \cup \Gamma_2, \Delta)\) is \(J\)-consistent. So, there is a model \((M, \theta)\) of this theory.

Claim. There are onto \(p\)-morphisms \(\{f_n\}\) from \(M\) to \(M_n\) such that \((M, \{f_n\})\) is a cone for \(S\).

Corollary 8.

1. The intuitionistic logic is the only strongly complete logic weaker than Gabbay-de Jongh’s logic of finite binary trees.
2. The intermediate propositional logics between Scott’s logic \(H + N_{10}\) and \(H + N_{10} + p \lor (p \supset (q \lor (q \lor (r \lor \neg r))))\) are not strongly complete.

The proof of the first result is a modification of Smoriński’s result that the infinite binary tree characterizes the intuitionistic propositional logic \(H ([7])\), and the second one is a modification of Rodenburg’s result that Scott’s axiom \(N_{10}\) is not elementary ([4]).

Since all known completeness proofs for intermediate predicate logics are in fact strong completeness proofs, the following problem is difficult to answer at the present time.

Problem 9. Find completeness proof applicable to finitely axiomatizable intermediate predicate logics whose propositional part is not strongly complete.

References


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