Troubles with modality in first-order Logic are widely known. Garson [4] presents useful topography of possible choices. The aim of this paper is not to discuss advantages and disadvantages of existing systems but to present their adequate ND-formulation.

Fitting’s book [1] is probably the most extensive study of various proof procedures for modal logic. Nevertheless, the problem of ND-systems for QML (Quantified Modal Logic) is only touched there. In this paper we consider 5 alternative approaches, being conservative extensions of standard QL. The main source of differences is the treatment of domains in models. All of them are well known from literature so we restrict description only to basic facts connected with ND-formalization.

1. Kalish/Montague ND-system for QL

ND-system is understood here in the strict sense as the one in which there are some rules for introducing and eliminating assumptions and analogous rules for logical constants (the latter are called inference rules). In this perspective neither Beth tableaux nor Gentzen’s sequent calculi belong to the category of ND-systems; the former lacks introduction rules for constants, the latter lacks elimination rules. The proof of the thesis in ND-system is in fact derivation of it from some assumptions, finally discharged. If derivation is linear (not a tree, like in original Gentzen’s ND-system) some additional technical devices, like numerals, lines, boxes etc. are neccessary in order to avoid invalid derivations.

Among many existing systems for QL, that of Kalish and Montague [6] seems to be especially well suited for our purposes. We can mention at least two profits: very simple rules for quantifiers and handy technical
devices necessary for dealing with restrictions connected with many non-classical logics. We only sketch the most important and specific properties of this system, one can find details in Kalish, Montague [6].

There are two different kinds of lines in their system:

a) **usable-lines**, which contain premises, assumptions or results of application of inference rules;

b) **show-lines** displaying formulas, which are to be proved; once one starts a derivation of a formula \( \Phi \), he enters the show-line of the form "Show: \( \Phi \)". Show-line can be turned onto usable-line by some **rule of closing a derivation**, specified bellow.

We slightly extend this terminology; every formula displayed on a show-line will be called **show-formula** and on any other, **usable-formula**. Other useful concept, which we add, is the notion of **degree of the derivation**, due to Gentzen. We can say that every show-line opens an \( i \)-degree derivation (\( i \geq 1 \)), when this show-line is closed, its show-formula becomes usable-formula in derivation of \( i - 1 \)-degree. Everytime we close show-line, the whole derivation of \( i \)-degree, being it’s justification becomes useless, so we close it in the box and cancel the prefix "Show:". Boxing-technique is due to Jaśkowski (see Prawitz [7]).

We can close a derivation of \( i \)-degree in four cases:

a) [Dir]: if \( \Phi \) is its show-formula and \( \Phi \) appears as usable-formula in \( i \)-degree derivation;

b) [Cond]: if \( \Phi \rightarrow \Psi \) is its show-formula and \( \Psi \) appears as usable-formula in \( i \)-degree derivation;

c) [Red]: if \( \bot \) appears as usable-formula in \( i \)-degree derivation (in this case the form of the show-formula has no importance);

d) [Univ]: if \( \forall \alpha \Phi \) is its show-formula and \( \Phi \) appeared as usable-formula in \( i \)-degree derivation, provided \( \alpha \notin VF(\Gamma) \), where:

\( -\Gamma \) is a set of all usable-formulas above show-line in question

\( -VF(\Gamma) \) is a set of free variables in \( \Gamma \)

Schematically we can show these rules as follows:
Assumptions may be entered only in connection with show-lines. There are two rules for starting assumptions:

a) if last show-line contains formula $\Phi \rightarrow \Psi$, we can add $\Phi$ as conditional assumption, immediately below this show-line;

b) if last show-line contains $\Phi$, we can add $\neg \Phi$ as indirect assumption, immediately below this show line.

Inference rules are as usual, only the rule of $\exists E$ needs some explanation. From usable-formula $\exists \alpha \Phi$ one can infer $\Phi[\alpha/\beta]$ (in short $\exists \alpha \Phi \vdash \Phi[\alpha/\beta]$), provided $\beta \notin V(\Gamma)$, where:

- $\Gamma$ is the set of all (usable and show) formulas, which appeared so far in the derivation;
- $V(\Gamma)$ is the set of all (free and bound) variables in $\Gamma$;
- $\Phi[\alpha/\beta]$ is the result of uniform instantiation of $\beta$ for every free occurrence of $\alpha$ in $\Phi$.

Remark 1. QL-rules were defined on languages that contain only individual variables but adding individual parameters and constants cause no troubles.
2. Modal Rules

For modal logic we need three rules of inference and an additional rule of closing a derivation:

a) $\Box \exists \Phi$ from $\exists \Phi$ infer $\Phi$ (not permissible in logic $K$);

b) $\Diamond l$ from $\neg \neg \neg \Phi$ infer $\Diamond \Phi$;

c) $\Diamond \exists \Phi$ from $\exists \Phi$ infer $\neg \neg \neg \Phi$;

d) [Necc]: We can close derivation of $i$-degree if $\Box \Phi$ is its show-formula and $\Phi$ appears as usable-formula in $i$-degree derivation, provided there is no indirect assumption $\neg \neg \neg \Phi$ under the show-line.

Unfortunately, this is not sufficient for modal logic. ND-formulations of many non-classical logics usually demand some devices, which forces to use some formulas in the derivation (like in relevance logic) or block the use of some other ones. The apparatus of hierarchy of derivations is well suited to do this job, because the only thing we need is to add the rule $R$ which transfers formulas from $i$-degree derivation to $i+1$-degree derivation. Before exact formulation of $R$ we list some groups of formulas.

Let $\Gamma$ be the a set of usable-formulas in some derivation then $\Gamma^*$ is defined relatively to a system of modal logic. Thus, e.g.

a) for $K$, $T$ $\Gamma^* = \{ \Phi : \Box \Phi \in \Gamma \}$;

b) for $S4$ $\Gamma^* = \{ \Box \Phi : \Box \Phi \in \Gamma \}$;

c) for $S5$ $\Gamma^* = \{ \text{Mod} \Phi : \text{Mod} \Phi \in \Gamma \}$, where $\text{Mod} \Phi$ is $\Box \Phi$ or $\Diamond \Phi$.

Remark 2. Consult Fitting [1] to see how the "characteristic" set $\Gamma^*$ may be constructed for other modal logics. The idea of modelling logics by the rule $R$ with restrictions on admissible formulas seems to be due to Fitch (see appendix one in [5]).

Now, the rule $R$: let $\Gamma$ be the set of $i$-degree usable-formulas. If $\Phi \in \Gamma$ then $\Phi$ can be added to $i+1$-degree derivation as a result of $R$ if

a) show-line opening $i+1$-degree derivation is not of the form "Show: $\Box \Psi"$

b) first usable-formula of $i+1$-degree derivation is indirect assumption. If neither a) nor b) is satisfied then $\Phi$ can be added, provided $\Phi \in \Gamma^*$.

Remark 3. This rather complicated form of $R$ and restriction added to [Necc] are not necessary. But such formulation enables us to get the same grade of flexibility in carrying derivations as in QL.
In order to facilitate derivations one can add another rule dealing with "\(\Diamond\)\(^n\), \(\Box \text{Red}\). It combines two rules, one for introducing new type of assumption and one for new type of closing derivations. In short:

If \(\{\Gamma, \Box \Phi\}\) is a set of \(i\)-degree derivation usable-formulas then we can start \(i + 1\)-degree derivation with "Show: \(\bot\)" and \(\Phi\) as its only assumption (mod. ass.). \(R\) is restricted to \(\Gamma^*\) (as in the case of closing by \([\text{Necc}]\)) and derivation is closed iff \(\bot\) appears as \(i + 1\)-degree derivation usable-formula.

**Remark 4.** Admissibility of \([\Diamond \text{Red}]\) is straightforward. It is based on the following metalogical law: If \(\Gamma^*, \Phi \vdash \bot\) then \(\Gamma, \Diamond \Phi \vdash \bot\), which is equivalent to: if \(\Gamma^* \vdash \neg \Phi\) then \(\Gamma \vdash \Box \neg \Phi\), but the latter is metalogical basis of (specific form of) \([\text{Necc}]\). In fact, we can go even further and generalise every rule of closing derivation to modal analogue with "mod. ass." and \(R\) restricted to \(\Gamma^*\), but there are needed some, not always easy describable, restrictions, so we omit this question here.

**Remark 5.** One can ask whether our original \(\Diamond I\) and \(\Diamond E\) rules cannot be replaced by \([\Diamond \text{Red}]\) and, for instance \(\Diamond I^l\): \(\Phi \vdash \Diamond \Phi\). Unfortunately, ND-system so obtained is incomplete. Neither \(\neg \Box \Phi \rightarrow \Diamond \neg \Phi\) nor \(\neg \Diamond \Phi \rightarrow \Box \neg \Phi\) is derivable, so at least \(\Diamond I\) is indispensable. Problem is analogous to incompleteness of Ohnishi/Matsumoto sequent systems for modal logics (see Routley [8]), and can be solved by introducing two additional constants (T and F) to language and rules. Complete ND-system with rules of this kind can be found in Fitting [1]).

### 3. ND-system for QML

All this formal machinery together with QL-rules is sufficient to get ND-system complete with respect to a given class of Kripke models, satisfying **Monotonicity condition** (MON) for domains:

If \(Rw1w2\) then \(D(w1) \subseteq D(w2)\), where \(R\) is an Accessibility relation, \(w1, w2\) are possible worlds and \(D(w)\) is the set of inhabitants of \(w\).

In this logic (QPL in Garson’s terminology) the converse of the Barcan Formula (CBF) is derivable but not the Barcan Formula (BF) itself. In order to get BF either modalities must be as strong as in S5 (which is
assured by proper definition of $\Gamma^*$) or, in the case of weaker (non symmetric) modalities, we must add new inference rule:

$$\forall\Box\text{-INT}(-\text{exchange}): \forall\alpha\Box\Phi \vdash \Box\forall\alpha\Phi$$

which directly enable to derive BF in every system. In both cases adequate semantics is represented by class of Kripke models with constant domain (Q1 in Garson).

In order to obtain ND-systems for QML with world-relative domains we must impose some conditions on derivations in our basic (without $\forall\Box\text{-INT}$) ND-system:

a) "Show: $\Phi$" can open derivation of 1-degree, provided $VF(\Phi) = \emptyset$;
b) "Show: $\Box\Phi$" can be closed by [Necc], provided $VF(\Phi) = \emptyset$;
c) "Show: $\Box\Phi$" can be closed by [Necc], provided $VF(\Phi) \subseteq VF(\Gamma^*)$;
d) let "Show: $\Box\Phi$" open $i+1$-degree derivation and $\Gamma^*$ be the suitable set of $i$-degree formulas then in the application of R, $\Gamma^*$ is replaced by $\Gamma^\# = \{\Psi \in \Gamma^*: VF(\Psi) = \emptyset\}$;
e) as d) above but $\Gamma^\# = \{\Psi \in \Gamma^*: VF(\Psi) \subseteq VF(\Phi)\}$.

The most widely known logic of this kind is Kripke's QK. It is also the most restrictive one, because only closed formulas from ordinary QL are valid in its semantics. Neither BF nor CBF is derivable, even for S5-like modalities. An equivalent ND-formulation requires addition of three conditions: a), b) and d).

ND-system with the only restriction c) is equivalent to Gabbay's logic GKc, where BF and CBF are still not present (untill we have S5-modalities), but all standard QL-basis is derivable. If we replace c) by e) it results in Gabbay's GKs. This last system is of special interest for those philosophers, who accept CBF but not BF, and prefer S5-modalities. It may be instructive to have a look at a derivation of BF in ND-QPL(S5) and to see why it doesn't work in ND-GKs(S5). The key point is in line 12, because the application of R(S5) violates restriction e).
Remark 6. Some other combinations of these restrictions may lead to different systems, which have no adequate semantics, but in some respects are similar to QK, GKc or GKS. For instance, in ND-QK we can omit condition a); BF and CBF are still not present but standard QL is derivable. ND-GKs with more restrictive condition d) instead of e) is unable to derive one of GKS axioms, namely version of K: $\Box(\Phi \rightarrow \Psi) \rightarrow (\Box\Phi \rightarrow \Box\Psi)$ with the restriction that $V F(\Phi) \subseteq V F(\Psi)$. But such a system is similar to GKS in this, that CBF is derivable, whereas BF is not.

4. Sketch of adequacy proof

There is no room here to give detailed proofs of adequacy for all of discussed systems. Soundness of presented rules is easy to establish. The only interesting case is to show that principles responsible for the form of $\text{[Necc]}$ and $R$ in every system are valid in every class of suitable models.
As an illustration we examine rules for GKc and GKS. Let us recall here some special features of semantics for these systems. Both logics satisfy TG (Truth Gap) condition:

if \( V(\alpha) \not\in D(w) \) then \( V(\Phi(\alpha),w) \) is undefined,

where \( V \) is an assignment function defined as in Hughes, Cresswell [5].

**Truth conditions** for "\( \Box \)" are the following:

- **GKc** \( V(\Box \Phi, w) = 1 \) iff if \( Rww' \), then \( V(\Phi, w') = \text{defined} \) and \( V(\Phi, w') = 1 \)
- **GKS** \( V(\Box \Phi, w) = 1 \) iff \( V(\Phi, w) = \text{defined} \) and if \( Rww' \) and \( V(\Phi, w') = \text{defined} \), then \( V(\Phi, w') = 1 \).

Remark 7. Clause for GKS is a bit different from original Gabbay’s definition (see Gabbay [2] or Garson [4]) but requirement that \( V(\Phi, w) = \text{defined} \) is necessary for \( \square \Phi \rightarrow \Phi \) to be valid in T and stronger logics (see Gamut [3]).

In GKc respective rules reflect the principle:

if \( \Gamma^* \models \Phi \) then \( \Gamma \models \Box \Phi \), where \( VF(\Phi) \subseteq VF(\Gamma^*) \).

Assume that \( \Gamma^* \models \Phi \) and that not \( \Gamma \models \Box \Phi \), then exists a model and a world in it, say \( w1 \), such that \( V(\Psi, w1) = 1 \), for every \( \Psi \in \Gamma \) and \( V(\Box \Phi, w1) = 0 \). By the truth-clause for "\( \Box \)" in GKc, there is a world \( w2 \) accessible from \( w1 \), such that either \( V(\Phi, w2) \) is undefined (i.e. some \( \alpha \) in \( \Phi \) has no denotation in \( D(w2) \) under some assignment \( V' \) being \( \alpha \)-variant of \( V \)) or \( V(\Phi, w2) = 0 \). By definition of \( \Gamma^* \), every \( \Psi \) in \( \Gamma \), which is also in \( \Gamma^* \) must be true in every world accessible from \( w1 \), so in \( w2 \), either. \( VF(\Phi) \subseteq VF(\Gamma^*) \) thus \( \Phi \) cannot be undefined in \( w2 \) and \( V(\Phi, w2) = 0 \), but, by assumption, if all members of \( \Gamma^* \) are true in \( w2 \) then \( V(\Phi, w2) \) must be 1. Q.E.D.

Proof that the principle: if \( \Gamma^\Phi \models \Phi \) then \( \Gamma \models \Box \Phi \), which justifies the respective rules in GKS, is valid in every class of models for GKS is similar.

Remark 8. Above mentioned semantic principles can serve also as a justification for sequent rules, in the style of Gentzen systems.

Completeness can be proved by some variant of Henkin method, described by Garson [4]. Particularly useful general scheme is extensively used by Fitting [1]. Essential feature of this technique is the application
of **Consistency Properties** (CON), which are not sets of formulas with desirable properties (like, for instance, Hintikka sets), but families of consistent sets of formulas (it means, of course, consistent with respect to some fixed logic).

Any CON satisfies the following conditions, for any Γ belonging to it:

a) ⊥ /∈ Γ;

b) if Φ ∈ Γ then ¬Φ /∈ Γ, for any Φ ∈ ATomic-formulas;

c) if (Φ ∧ Ψ) ∈ Γ then Γ ∪ {Φ, Ψ} ∈ CON;

d) if ¬(Φ ∧ Ψ) ∈ Γ then Γ ∪ {¬Φ} ∈ CON or Γ ∪ {¬Ψ} ∈ CON;

e) if ∀αΦ ∈ Γ then Γ ∪ {Φ[α/τ]} ∈ CON, for any term τ;

f) if ¬∀αΦ ∈ Γ then Γ ∪ {¬Φ[α/τ]} ∈ CON, for some term τ;

g) if □Φ ∈ Γ then Γ ∪ {Φ} ∈ CON;

h) if ¬□Φ ∈ Γ then Γ* ∪ {¬Φ} ∈ CON.

**Remark 9.** Conditions for other constans are analogous, as can be expected.

**Remark 10.** The above characteristic is too general to cover all the cases. In fact it is insufficient for logics with S5-like modalities (but see Fitting [1]). Also in condition h) Γ* must be replaced with Γ# when we consider QK and GKs. On the other hand condition g) must be omitted in the case of K-like modalities.

Fitting shows in detail how we can construct maximal extension for any Γ ∈ CON and then define a model satisfying all formulas in Γ, so we limit ourselves to the first step; we show that the set of all Γ such that not Γ ⊢ ⊥ is CON (in fact, we show only two cases for the sake of illustration).

For d) assume that, not Γ, ¬(Φ ∧ Ψ) ⊢ ⊥ and (indirectly) that Γ, ¬Φ ⊢ ⊥ and Γ, ¬Ψ ⊢ ⊥, then we can construct ND-derivation Γ, ¬(Φ ∧ Ψ) ⊢ ⊥, contrary to our assumption. For h) assume that, not Γ, ¬□Φ ⊢ ⊥, and (indirectly) that Γ*, ¬Φ ⊢ ⊥, then we can construct ND-derivation Γ, ¬□Φ ⊢ ⊥, contrary to our assumption. The following schemes display essential points of such derivations. We can see how to make a suitable embedding of one derivation in another.
Show: ⊥  [3,12,Red]

Γ  

¬(Φ ∧ Ψ)  ass.

Show: Φ  [7,Red]

¬Φ  ind. ass.

Γ 2,R

⊥  5,6, by ass.

Show: Ψ  [11,Red]

¬Ψ  ind. ass.

Γ 2, R

⊥  9,10, by ass.

Φ ∧ Ψ  4,8, ∧I

Show: ⊥  [3,4,Red]

Γ  ass.

¬□Φ  ass.

Show: □Φ  [6,Necc]

¬Φ  2,R

Γ*  [9,Red]

ind. ass.

Γ*  5, R

⊥  7,8, by ass.

39
Having shown that the set of all admissible ND-derivations is CON, we know that whenever Φ is not ND-derivable, \{¬Φ\} ∈ CON. It means that there exist a model and a world w in it such that V(¬Φ, w) = 1, and so Φ is invalid. In fact, even strong completeness follows if a logic under consideration is compact.

References


Department of Logic
Lódz University
Matejki 34a
90-237 Lódz
Poland