ON THE NATURAL ORDER RELATION
IN PEANO ALGEBRAS
WITH FINITARY OR INFINITARY OPERATIONS

Most work on infinitary algebras in the literature makes ample use of
the axiom of choice AC (see, e.g., [10],[12],[14]). While the theory of arbitrary
infinitary algebras can hardly be developed in a satisfactory manner
without AC, in the case of Peano algebras (= word algebras, absolutely
free algebras)\(^1\) many or even most of the basic results can be proved in
Zermelo-Fraenkel set theory ZF without the axiom of choice, sometimes,
however, only with considerable effort.

A convenient tool for proving structure theorems as well as classifying
Peano algebras \(A\)—e.g., according to the number of operations and their
arities—is the algebraic predecessor relation \(P_A\) resp. its transitive hull \(<_A\),
the natural ordering of \(A\). This relation, which is well-founded in every
Peano algebra with even arbitrarily many infinitary operations (see [1],[2]),
was used in [4] for characterizing several classes of Peano algebras. There,
at one point, however, the countable axiom of choice \(AC\) was applied to
obtain, for a given operation \(f_i\) of infinite arity \(K_i\), a countable subset of \(K_i\)
(see [4], Proposition 5.6). Hence, infinite Dedekind-finite arities were
excluded.

In the following we will fill in this gap. Without using any form of the
axiom of choice we will determine those Peano algebras whose natural order
relation is a jump ordering. It will turn out that they can be characterized
as the Peano algebras \(A\) all elements of which have finite Baire rank
(over basis \(A_0\)), or—using the notion of a surjective Hartogs function
introduced in [3]—as the Peano algebras where all operations are finitary
or have infinite Dedekind-finite arities of surjective Hartogs value \(\aleph_0\).

\(^1\)The notion of a Peano algebra resp. of an absolutely free algebra appeared first
in Mathematical Logic: The algebra of terms and the algebra of formulas of a formal
language are absolutely free algebras.

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The proof is independent from that of Proposition 5.6 in [4], yielding a more general result.

1. Notation and Terminology. Basic Facts

We will work in one of the familiar systems of predicative set theory without the axioms of choice and foundation, e.g., in ZF$^-$. All objects we will deal with here are "pure" sets. We use customary set-theoretical and algebraic notation (see, e.g., [5],[9] resp. [6],[7]). For the reader’s convenience, however, some particular notions and statements used below will be listed here.

For any set $x$, $\aleph(x)$ denotes the usual (injective) Hartogs value of $x$. This Hartogs function is, however, not suited for our purposes (see Remark 3.5). As a counterpart to the injective Hartogs value we define, for any set $x$, the surjective Hartogs number $\aleph^*(x)$ to be the class

\[(1.1) \quad \aleph^*(x) := \{\alpha \in \text{ON} | \alpha = 0 \lor \text{there is a surjection from } x \text{ onto } \alpha\}.
\]

$\aleph^*(x)$ is a transitive set and therefore an ordinal. As a matter of fact, it is the least ordinal which is not a surjective image of $x$, hence a (well-ordered) cardinal.

Clearly, we have $\aleph(x) \leq \aleph^*(x) \leq \aleph(\mathcal{P}(x))$ for every set $x$ (where $\mathcal{P}(x)$ denotes the power-set of $x$), and $\aleph(x) = \aleph^*(x) = |x|^+$ in case $x$ can be well-ordered.

Let us recall that set $x$ is Dedekind-finite, if there is no bijection from $x$ onto a proper subset of $x$. Otherwise, $x$ is called Dedekind-infinite or transfinite. $x$ is Dedekind-infinite if and only if it includes a countable subset. An infinite Dedekind-finite set is called a Dedekind set, its cardinal a Dedekind cardinal.

2. Relations and Algebras

For any two sets $A$ and $B$, $A \setminus B$ denotes the set-theoretical difference, $\omega = \omega_0$ is the least infinite ordinal, and $\text{ON}$ is the class of all von Neumann ordinals.

Given an arbitrary relation $R$, i.e., a set of ordered pairs $(a, b)$, $R^{-1}$ is the converse relation, and $R[X]$ denotes the $R$-image of set $X$. If $R$ is
a relation and \( a R b \) holds, then we call \( a \) an \( R \)-predecessor of \( b \), and \( b \) an \( R \)-successor of \( a \).

Let \( \langle A; R \rangle \) be a (binary) structure, i.e., a set \( A \) together with a relation \( R \subseteq A \times A \). \( B \subseteq A \) is an initial \( R \)-segment of \( A \) if \( R^{-1}[B] \subseteq B \). The closure operator assigning to every subset \( M \) of \( A \) the least initial \( R \)-segment including \( M \) will be denoted by \( C_R \). An initial \( R \)-segment \( B \) is called a principal initial \( R \)-segment if \( B = C_R \{ x \} \) for some \( x \in A \).

For any relation \( R \) on \( A \), the set of all transitive relations on \( A \) is a closure system. Hence, there exists, for every relation \( R \) on \( A \), the smallest transitive relation including \( R \), the transitive hull \( R^* \) of \( R \). \( R^* \) is also called the ancestor relation of \( R \).

For any relation \( R \) on \( A \) and all \( x, z \in A \), the following holds:

\[
(2.1) \quad z R^* x \iff \exists y (y R x \land z \in C_R \{ y \}).
\]

A type for algebras is any set-valued function \( \kappa = (K_i)_{i \in I} \) on a set \( I \). An algebra of type \( \kappa \) (in short: \( \kappa \)-algebra) is an ordered pair \( A = \langle A; (f_i)_{i \in I} \rangle \) where \( (f_i)_{i \in I} \) is an \( I \)-family of \( K_i \)-ary operations \( f_i \) on set \( A \), i.e., of mappings \( f_i : A^{K_i} \to A \).

As usual, we identify algebra \( A \) and its universe \( A \), if no confusion can arise. The closure system of all subalgebras (more precisely: subuniverses) of \( A \) will be denoted by \( Su(A) \), and the corresponding closure operator, assigning to every set \( M \subseteq A \) the smallest subalgebra including \( M \), by \( Sg(A) \). Closure induction with respect to the closure system \( Su(A) \) of all subalgebras of algebra \( A \) is also called “algebraic induction” in the literature. We have \( Sg(A)^M = \bigcup_{\alpha \in \text{ON}} B^{\alpha}(M) \), where \( B^{\alpha}(M) \) is the \( \alpha \)-th Baire class over \( M \). Since \( A \) is a set, so is \( Sg(A)^M \); hence there exists an ordinal \( \gamma \) with \( Sg(A)^M = \bigcup_{\alpha \in \gamma} B^{\alpha}(M) \). For any \( x \in Sg(A)^M \), the Baire rank \( rk(x) \) of element \( x \) over set \( M \) is the least ordinal \( \alpha \) with \( x \in B^{\alpha}(M) \).

For any algebra \( A = \langle A; (f_i)_{i \in I} \rangle \), the set of all constants of \( A \) (= nullary operations) is denoted by \( A^c \), and the set of all elements which are not values of any of the operations, by \( A_0 \).

**Definition 2.2.** An algebra \( A = \langle A; (f_i)_{i \in I} \rangle \) of type \( \kappa = (K_i)_{i \in I} \) is a Peano algebra, if the following “generalized Peano axioms” hold ([1]):

\[ P1. \text{ all operations } f_i \text{ are injective,} \]
P2. the ranges of the operations \( f_i \) are pairwise disjoint,

P3. the set \( \mathcal{A}_0 \) of all elements which are not values of any of the operations \( f_i \) form a generating subset (induction axiom).

The axioms P1, P2, P3 are a direct generalization of the Dedekind-Peano axioms for the natural numbers, i.e., in case of just one unary operation \( \ast \) and a one-element \( \{0\} \) generating subset, the axiom system P1, P2, P3 reduces to the classical Peano-Dedekind axioms for the natural numbers. It is well-known that the Peano algebras are just the absolutely free algebras, i.e., the free algebras in the class of all algebras of same type \( \kappa \) (see, e.g., [10],[14],[1],[13]).

**Definition 2.3.** Let \( \mathcal{A} = \langle \mathcal{A}; (f_i)_{i \in I} \rangle \) be an algebra of type \( \kappa = (K_i)_{i \in I} \).

We define on \( \mathcal{A} \) the algebraic predecessor relation \( \mathcal{P}_\mathcal{A} \) by

\[
(2.4) \quad bP_\mathcal{A} a := (\exists i \in I)(\exists a \in A^{K_i})(\exists k \in K_i)(b = a(k) \land a = f_i(a)).
\]

We call \( b \) an algebraic predecessor of \( a \), or \( a \) an algebraic successor of \( b \). Not much can be said about the algebraic predecessor relation \( \mathcal{P}_\mathcal{A} \) in an arbitrary algebra \( \mathcal{A} \). For Peano algebras, however, we have the following fundamental

**Proposition 2.5.** (see [1],[2]). For any Peano algebra \( \mathcal{A} \), the algebraic predecessor relation \( \mathcal{P}_\mathcal{A} \) is well-founded, hence its transitive hull \( \mathcal{P}_\mathcal{A}^* \) is a well-founded irreflexive partial ordering.

If, for a given algebra \( \mathcal{A} \), relation \( \mathcal{P}_\mathcal{A}^* \) is an irreflexive partial ordering, we will call \( \mathcal{P}_\mathcal{A}^* \) the natural ordering or the natural order relation of \( \mathcal{A} \), and also use the notation \( \prec_\mathcal{A} \) or simply \( \prec \), if no confusion can arise; \( \leq_\mathcal{A} \) resp. \( \leq \) will denote the corresponding reflexive partial order.

3. The Theorem

Without using any form of the axiom of choice we will characterize here the class of all Peano algebras \( \mathcal{A} \) whose operations \( f_i \) have arities \( K_i \) of surjective Hartogs value \( \leq \aleph_0 \). We need two more notions, well-known in the theory of relations (see, e.g., [11]).

**Definition 3.1.** Let \( R \) be an irreflexive partial ordering. We define:
Thus $a \prec_R b$ (“$a$ is covered by $b$”, “$a$ is a lower neighbor of $b$”) means that we have $a R b$, but there is no element between $a$ and $b$. Relation $\prec_R$ is called the covering or neighborhood relation of $R$.

(2) $R$ is a jump ordering if it is the transitive hull of its neighborhood relation, i.e., if $R = (\prec_R)^*$.

An ordered pair $\langle a, b \rangle \in R$ with $a \prec_R b$ is called a jump (or a gap), and relation $R$ is a jump ordering just in case it is completely determined by its jumps, $R$ being the transitive hull of its neighborhood relation. If $\mathcal{A} = \langle A; (f_i)_{i \in I} \rangle$ is a Peano algebra with natural order $<_A$ we will denote the neighborhood relation in short by $\prec_A$ rather than by $\prec_{(\langle A \rangle)}$.

**Theorem 3.2.** Let $\mathcal{A} = \langle A; (f_i)_{i \in I} \rangle$ be a non-empty Peano algebra of arbitrary type $\kappa = (K_i)_{i \in I}$. Then the following are equivalent:

1. $\aleph^*(K_i) \leq \aleph_0$ for all $i \in I$;
2. $<_A$ is a jump ordering;
3. $A = \bigcup_{k \in \omega} B^k(A_0)$, i.e., every $x \in A$ has finite Baire rank over $A_0$.

**Proof.** (1)$\rightarrow$(3). This implication holds for every algebra whatsoever. Proof by algebraic induction on $x$.

(2)$\rightarrow$(1). Assume that there exists a $K_i$-ary operation $f_i$ with $\aleph^*(K_i) > \aleph_0$. We define two secondary operations $g$ and $h$: $g$ being unary and $h$ having arity $\aleph_0$. We obtain $g$ from $f_i$ by “identifying” all arguments, i.e., for every $x \in A$, we stipulate $g(x) := f_i(a_k)_{k \in K_i}$ with $a_k = x$ for all $k \in K_i$. Furthermore, let $\varphi : K_i \sim \aleph_0$ be a fixed surjective mapping from $K_i$ onto $\aleph_0$ (which exists by assumption), and let $(b_n)_{n \in \omega}$ be an arbitrary $\omega$-sequence of elements in $A$. We define

$$h(b_n)_{n \in \omega} := f_i(a_k)_{k \in K_i},$$

with $a_k := b_n$ for all $n \in \omega$ and $k \in \varphi^{-1}\{n\}$. Now, let $x_0$ be an arbitrary element of $A$. Clearly, the subalgebra $B$ of the unary algebra $\langle A; g \rangle$ generated by $B_0 := \{x_0\}$ satisfies the Dedekind-Peano axioms for the natural numbers (with $g$ as successor function and $x_0$ as zero), hence its universe $B$ is a countable set. Let $b = (b_n)_{n \in \omega}$ be a bijection from $\omega$ onto $B$. For any natural number $n \in \omega$ we have $b_n <_A h(b)$. Moreover, since we have assumed that $<_A$ is a jump ordering, i.e., that $<_A = (<_A)^*$, (2.1) implies
the existence of an element \( b' \) with \( b_n \leq A b' \prec_A h(b) \), hence \( b_n P_A h(b) \). But then, by the Peano axioms P1, P2, \( b' = b_m \) for some \( m \in \omega \). Hence \( b' = b_m <_A g(b_m) <_A h(b) \), contradicting \( b' \prec_A h(b) \). □

(3) → (2). We can reformulate (2) as follows

\[
(3.3) \quad a <_A b \rightarrow \exists b'(b' \prec_A b \land a \in C <_A \{b'\}).
\]

Assume now that \( a <_A b \), and let \( \Gamma \) be the following (non-empty) set

\[
\Gamma := \{rk(b') \mid a \leq A b' \land b' P_A b\}.
\]

Since \( P_A \) is well-founded ([2]), \( b' P_A b \) implies \( rk(b') < rk(b) \), and we have \( rk(b) = \sup^+ \{rk(b') \mid b' P_A b\} \). But \( rk(b) \in \omega \), by assumption, hence \( \Theta := \{rk(b') \mid b' P_A b\} \) is a bounded (= finite) set of natural numbers including \( \Gamma \). Therefore, \( \Gamma \) contains a maximum \( t \) (with respect to the order relation \( <_N \) in the natural number system \( \mathbb{N} \)). We choose an element \( b^* \) with \( a \leq A b^* \), \( b^* P_A b \), and \( rk(b^*) = t \). Clearly, \( b^* \prec_A b \); hence we have proved the following implication

\[
(3.4) \quad a <_A b \rightarrow \exists b^*(b^* \prec_A b \land a \leq_A b^*).
\]

Since \( a \leq_A b^* \) is equivalent to \( a \in C_{P_A}\{b^*\} \), it remains to be shown that \( C_{P_A}\{x\} = C_{<_A}\{x\} \) for every \( x \in A \). Clearly, \( C_{P_A}\{x\} \supseteq C_{<_A}\{x\} \) for every \( x \in A \). We prove the converse \( C_{P_A}\{x\} \subseteq C_{<_A}\{x\} \) by algebraic induction on \( x \):

I. For every \( x \in A_0 \) we have \( C_{P_A}\{x\} = C_{<_A}\{x\} = \{x\} \).

II. Let \( x = f_i(a) \) for some \( i \in I, a \in A^K_i \). Assume \( a \in C_{P_A}\{x\} \).

If \( a = x \), then trivially \( a \in C_{<_A}\{x\} \). If \( a \neq x \), i.e., \( a <_A x \), then there exists, by (2.1), an element \( b^* \) with \( b^* \prec_A x \) and \( a \in C_{P_A}\{b^*\} \). But then \( a \in C_{<_A}\{b^*\} \), by the induction hypothesis and the Peano axioms, hence also \( a \in C_{<_A}\{x\} \). □

Remark 3.5. Even in Peano algebras with infinitary operations \( f_i \) the natural ordering can be a jump ordering and every element can be of finite Baire rank: We choose an arity \( K_i := d \) where \( d \) is a Dedekind set such
that $P(d)$ is also a Dedekind set. Since every Dedekind set has injective Hartogs value $\aleph_0$, we obtain $\aleph_0 = \aleph(d) \leq \aleph^*(d) \leq \aleph(P(d)) = \aleph_0$, hence $\aleph^*(d) = \aleph_0$. □

On the other hand, there are—in certain models of ZF—Peano algebras with operations $f_i$ of Dedekind-finite arity $K_i$ containing elements of infinite rank: What we need is a Dedekind-finite set $x$ with a surjective Hartogs number $> \aleph_0$. But every model of set theory which contains a Dedekind set, also contains $2^{\aleph_0}$-many Dedekind sets with surjective Hartogs values $> \aleph_0$. This follows from a construction due to Tarski [15]: For any subset $S$ of $\omega$ with $S \setminus \{\emptyset\} \neq \emptyset$, if $d$ is a Dedekind set, so is the set $\bigcup_{n \in S} d^{(n)}$ where $d^{(n)}$ is the set of all injections from $n$ into $d$. Obviously, for every infinite subset $T \subseteq \omega$, $b = \bigcup_{n \in T} d^{(n)}$ is a Dedekind set with surjective Hartogs value $\aleph^*(b) > \aleph_0$. Hence, any Peano algebra $A$ with such a $b$-ary operation $f_i$ contains elements of rank $\geq \aleph_0$ over basis $A_0$.

It follows from the foregoing that no condition for the injective Hartogs value of arity $K_i$ could characterize Peano algebras with jump orderings, or Peano algebras which have only elements of finite Baire rank. The natural ordering of a Peano algebra $A$ with an operation $f_i$ whose arity $K_i$ has an injective Hartogs value $\aleph_0$ could be or could not be a jump ordering. Likewise, there could be or could not be elements of infinite rank.

References


2It is well-known that this assumption, added to the axioms of ZF, yields a consistent theory, provided ZF is consistent (see, e.g., [8]).

3One can prove more: If there is an element $x \in A$ of infinite rank, then there exist elements of arbitrary countable Baire rank $\varrho$ over basis $A_0$. This follows from a result proved by the author in a forthcoming paper: On rank and dimension in infinitary Universal Algebra without choice.


Mathematisches Institut
Universität zu Köln
Weyertal 86-90
5000 Köln 41, Germany

E-mail: diener@mi.Uni-Koeln.DE