Kripke frame semantics is used as a concise and convenient apparatus for the study of non-classical predicate logics, but it is well-known that there exist a great many important logics which cannot be complete with respect to this semantics. Logicians have been introducing many kinds of new semantics to get rid of this difficulty. (cf. [1], [3]) Hence it is important to find a non-trivial example of the logics complete with respect to these new semantics.

The second author [4] found finitely axiomatizable and Kripke-frame incomplete super-intuitionistic predicate logics each of which is complete with respect to Kripke sheaf semantics, which is an extended version of Kripke frame semantics introduced by V. B. Shehtman and D. P. Skvortsov [3].

In this article, we discuss a generalization of this result.

By a careful reading of [4], the proof turns out to consist of the following two parts:

- the strong completeness of equational theories over a given intermediate predicate logic,
- the correspondence between equational theories and super-intuitionistic predicate logics as the logical fragments of equational theories.

Let us describe this fact more precisely.

We use the terminologies and notations as in [4]. For each intermediate predicate logic $L$, we denote by $L^\varepsilon$ the theory obtained from $L$ by adding the equality axioms. Let $EQ\ (EQ^+)$ be the set of all sentences (i.e. formulas without free occurrences of variables) containing no occurrences of
predicate symbols other than (positive occurrences of, respectively) equality =. For every subset $T \subseteq EQ$, the logical fragment $L_T$, of the theory $L^w + T$ is the set of formulas containing no occurrences of = and provable in $L^w + T$. Clearly, every logical fragment $L_T$ is a super-intuitionistic predicate logic.

**Theorem 1.**

1. Let $T$ be a subset of $EQ$. If an intermediate predicate logic $L$ is strongly complete with respect to Kripke frame semantics, then $L_T$ is strongly complete with respect to Kripke sheaf semantics.
2. Let $T$ be a finite subset of $EQ$. If an intermediate predicate logic $L$ is complete with respect to Kripke frame semantics, then $L_T$ is complete with respect to Kripke sheaf semantics.

**Proof.** (1) Suppose that a pair $(\Gamma, \Delta)$ of formulas is $L_T$-consistent. Without loss of generality, we may assume that both $\Gamma$ and $\Delta$ are sets of sentences. Let $\Sigma$ be the set of sentences provable in the theory $L^w + T$. It is obvious that $(\Gamma \cup \Sigma, \Delta)$ is $L$-consistent. Since $L$ is strongly complete with respect to Kripke frame semantics, there exist a Kripke frame $\langle M, U \rangle$ and a valuation $|=\,$ on it such that

- $\langle M, U \rangle$ validates $L$,
- for every sentence $A \in \Gamma \cup \Sigma \cup \Delta$, $0_{M} \models A$ if and only if $A \in \Gamma \cup \Sigma$.

where $0_{M}$ is the least element of the underlying poset $M = \langle M, \leq \rangle$. For each $a \in M$, let $\theta_a$ be the restriction of the interpretation of = to the domain $U(a)$ at $a$, that is,

$$u \theta_a v \text{ if and only if } a \models \pi = \bar{\pi},$$

for every $u, v \in U(a)$, where $\pi$ and $\bar{\pi}$ are the names of $u$ and $v$, respectively. Clearly, each $\theta_a$ is an equivalence relation on $U(a)$. Let $D$ be the set $\{ (a, u/\theta_a); a \in M \text{ and } u \in U(a) \}$. We define a quasi-order $R$ on $D$ and a mapping $\pi$ from $D$ to $M$ by

$$(a, u/\theta_a)R(b, v/\theta_b) \text{ if and only if } a \leq b \text{ and } u \theta_b v,$$

$$\pi((a, u/\theta_a)) = a,$$
for every \((a, u/\theta_a), (b, v/\theta_b) \in D\). Then, the triple \((D, M, \pi)\) is a Kripke sheaf, where \(D = (D, R)\). It is a routine to check that the following valuation \(\models^*\) is well-defined:

\[
a \models^* p((a, u_1/\theta_a), \ldots, (a, u_n/\theta_a)) \text{ if and only if } a \models p(u_1, \ldots, u_n),
\]

for every element \(a \in M\), every \(n\)-ary predicate symbol \(p\), and every \((a, u_1/\theta_a), \ldots, (a, u_n/\theta_a) \in \pi^{-1}(a)\). By induction, we have

\[
a \models^* A \text{ if and only if } a \models A
\]

for every \(a \in M\), every sentence \(A \in \Gamma \cup \Sigma \cup \Delta\). Hence,

- for every sentence \(A \in \Gamma \cup \Delta\), \(0_M \models^* A\) if and only if \(A \in \Gamma\).

It remains to show the soundness of \(L_T\) with respect to this Kripke sheaf. Define a set \(\tilde{U}\), a relation \(\rho\) on \(\tilde{U}\) and a mapping \(\pi\) from \(\tilde{U}\) to \(M\) by

\[
\tilde{U} = \{(a, u); a \in M \text{ and } u \in U(a)\},
\]

\[(a, u) \rho (b, v) \text{ if and only if } u = v \text{ and } a \leq b, \quad \pi((a, u)) = a.\]

We can show that \((\tilde{U}, \rho, M, \pi)\) is a Kripke sheaf, and that valuations on \((M, U)\) are valuations on \((\tilde{U}, \rho, M, \pi)\) as a Kripke sheaf, and vice versa. Moreover, the mapping \(f\) from \(\tilde{U}\) to \(D\) defined by

\[
f((a, u)) = (a, u/\theta_a)
\]

is a \(p\)-morphism introduced in [4]. Hence \(L\) is valid under any valuation on \((D, M, \pi)\).

Suppose a valuation \(\models^\circ\) on \((D, M, \pi)\) is given. We extend \(\models^\circ\) by adding the interpretation of \(=\) as:

\[
a \models^\circ (a, u/\theta_a) = (a, v/\theta_a) \text{ if and only if } (a, u/\theta_a) = (a, v/\theta_a)
\]

for every \(a \in M\) and every \((a, u/\theta_a), (a, v/\theta_a) \in \pi^{-1}(a)\). It is easy to see that all equality axioms are valid in this Kripke sheaf model \((D, M, \pi, \models^\circ)\).

Moreover, since the interpretation of \(=\) in \(\models^\circ\) coincides with that in \(\models^*\), \(a \models^\circ A\) holds for every \(a \in M\) and every sentence \(A \in T\). (Recall that \(A\) contains no occurrences of predicate symbols other than \(=\).)

Therefore \(L^\circ + T\) is valid in \((D, M, \pi, \models^\circ)\). Since \(\models^\circ\) is arbitrary, \(L_T\) is sound with respect to \((D, M, \pi)\).
Let \( A \) be a sentence which is not provable in \( L_T \), and \( \Sigma' \) be the set of the equality axioms containing the predicate symbols occurring in \( A \). Replace \( \Gamma, \Delta \) and \( \Sigma \) by \( \emptyset, \{ A \} \) and \( T \cup \Sigma' \), respectively in the proof of (1). Since \( T \cup \Sigma' \) is finite, the proof of (1) works in this case. \( \text{q.e.d.} \)

For every formula \( A \in EQ \), we denote by \( A^p \) the formula obtained from \( A \) by replacing each occurrences of \( s = t \) by \( \forall x(p(s, x) \equiv p(t, x)) \). For every \( T \subseteq EQ \), define
\[
T^p = \{ A^p; A \in T \}.
\]

**Theorem 2.** If \( T, T' \subseteq EQ^+ \), then \( L^w + T = L^w + T' \) if and only if \( L_T = L_{T'} \).

**Proof.** For each \( A \in EQ \), we denote by \( A^w \) the formula obtained by substituting \( = \) for each occurrence of \( p \) in \( A^p \). Then we can easily verify the following for each \( A \in EQ^+ \):
1. \( H^w \vdash A \supset A^w \), where \( H \) denotes the intuitionistic predicate logic,
2. \( L^w + T \vdash A^w \) implies \( L^w + T \vdash A^w \) for every \( T \subseteq EQ \),
3. \( H^w \vdash A \equiv A^w \).

Suppose \( L_T = L_{T'} \). Then \( L^w + T' \vdash A^p \) holds for every \( A \in T \) by 1. Hence \( L^w + T' \vdash A \) holds for every \( A \in T \) by 2 and 3. Therefore \( L^w + T' \supseteq L^w + T \). \( L^w + T \supseteq L^w + T' \) is proved similarly. \( \text{q.e.d.} \)

**Remark.** We cannot replace \( EQ^+ \) by \( EQ \) in Theorem 2. For example, the logical fragment of the intuitionistic theory of the decidable equality \( H^w + \forall x \forall y(x = y \lor x \neq y) \) remains \( H \).

**Theorem 3.** If \( T \subseteq EQ^+ \), then the logic \( L^w + T^p \) obtained from \( L \) by adding \( T^p \) as the logical axioms equals the logic \( L_T \).

**Proof.** The proof of Theorem 2 tells us that \( L_T \vdash A^p \) holds for \( A \in T \subseteq EQ^+ \). Hence \( L_T \supseteq L^w + T^p \). So it suffices to show the converse inclusion \( L^w + T^p \supseteq L_T \).

Suppose \( L_T \vdash B \). Then there exist finitely many predicate symbols \( p_1, \ldots, p_m \) and \( A_1, \ldots, A_n \in T \) such that the following holds:
\[
L \vdash E(p_1) \land \cdots \land E(p_m) \land A_1 \land \cdots \land A_n \supset B,
\]

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where $E(p_i) \ (i = 1, \ldots, m)$ denotes the equality axiom for $p_i$.

Let $E(p_i)' \ (i = 1, \ldots, m)$ ($A_j' \ (j = 1, \ldots, n)$, respectively) be the formula obtained from $E(p_i)$ ($A_j$, respectively) by replacing each occurrence of $s = t$ by the following formula:

$$\forall x_1 \bigwedge_{p_i} \forall x_1, \ldots, x_{k-1}, t, x_{k+1}, \ldots, x_{l_i} \left( p_i(x_1, \ldots, x_{k-1}, s, x_{k+1}, \ldots, x_{l_i}) \equiv p_i(x_1, \ldots, x_{k-1}, t, x_{k+1}, \ldots, x_{l_i}) \right),$$

where $l_i$ is the arity of $p_i$. Note that the above formula is equivalent to the following in $H$:

$$\forall x_1 \bigwedge_{p_i} \forall x_1, \ldots, x_{k-1}, t, x_{k+1}, \ldots, x_{l_i} \left( p_i(x_1, \ldots, x_{k-1}, t, x_{k+1}, \ldots, x_{l_i}) \equiv p_i(x_1, \ldots, x_{k-1}, s, x_{k+1}, \ldots, x_{l_i}) \right) \equiv$$

Clearly $H \vdash E(p_i)'$ for every $p_i \ (i = 1, \ldots, m)$ and $A_j' \ (j = 1, \ldots, n)$ is a substitution instance of $A_j'$. Since $L$ is closed under substitution, $L \vdash A_1' \land \cdots \land A_n' \supset B$ holds. Therefore $L + T \vdash B$.

It follows from the above two theorems that the study of a super-intuitionistic predicate logic of the form $L + T$ for some $T \subseteq EQ^+$ is equivalent to that of the theory $L^+ + T$.

In the following we list some examples under the conditions that $L$ is the intuitionistic predicate logic $H$ and $T$ consists of only one formula.

1. If $T = \{ \neg \forall x \forall y (x = y) \}$, $H_T$ is just the same one in Suzuki [4].
2. If $T = \{ \forall x \forall y \neg (x = y) \}$, $H_T$ differs from examples in Suzuki [4].
3. If $T = \{ \forall x \forall y \forall z (x = y \lor y = z \lor z = x) \}$, $H_T$ is complete with respect to the class of all Kripke sheaves with domains having at most two elements.
4. If $T = \{ \forall x \exists y (x = y \lor \forall z (y = z \lor z = x)) \}$, $H_T$ is complete with respect to the class of all Kripke frames with domains having at most two elements.

5. If $T = \{ \exists x \exists y \forall z (z = x \lor z = y) \}$, $H_T$ is complete with respect to the class of all Kripke frames with domains having just two elements.

Finally, we mention a preservation result on the propositional part of $L_T$ (i.e. the set of propositional formulas provable in $L_T$).

Corollary 4. For every $T \subseteq EQ^+$, the propositional part of $L_T$ is equal to that of $L$.

Proof. Since, for every $A \in EQ^+$, $\forall x \forall y (x = y) \supset A$ belongs to $H^\omega$, it suffices to show the case $T = \{ \forall x \forall y (x = y) \}$. By Theorem 3, we can easily show that $L_T$ is the strongest super-intuitionistic predicate logic $L + \forall x \forall y (p(x) \equiv p(y))$ whose propositional part is equal to that of $L$ (cf. Ono [2], §6). q.e.d.

Remark. In general, the propositional part of $L_T$ for $T \subseteq EQ$ can be stronger than that of $L$. Let $L$ be the strongest intermediate predicate logic whose propositional part is the intuitionistic logic and $T = \{ \exists x \exists y (x \neq y) \}$. Then $L_T$ becomes the classical logic, since $p \lor \neg p \lor \forall x \forall y (x = y)$ is in $L$.

We can give a semantical proof of this fact. $L$ is characterized by the set of Kripke frames containing only one element in each domain and the Kripke frames for the classical logic. Since $L^\omega + T$ is not satisfiable in the former types of Kripke frames, $L_T$ is characterized by the frames for the classical logic by Theorem 1.

In contrast to this, for every intermediate logic $L$ and every $T \subseteq EQ^+$, $L^\omega + T$ is satisfiable in any Kripke frame for $L$ under some suitable valuation, which gives a semantical explanation of Corollary 4.

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