CONSTRUCTION OF CLASSICAL PROPOSITIONAL LOGIC

Abstract

In this paper we give an independent axiomatization of the (full) classical propositional logic $TV$ by means of 7 implicational axioms. It allows to contract 7-dimensional cube, which we call $TV$-construction. Some properties of $TV$-construction are investigated.

In [5] and [6] the problem of extending $H_→$ to $TV_→$ was raised, where $H_→$ is an implicational fragment of intuitionistic propositional logic:

\begin{align*}
\text{I.} & \quad p \to p \\
\text{B.} & \quad (q \to r) \to ((p \to q) \to (p \to r)) \\
\text{C.} & \quad (p \to (q \to r)) \to (q \to (p \to r)) \\
\text{W.} & \quad (p \to (p \to q)) \to (p \to q) \\
\text{K}_1' & \quad r \to ((p \to q) \to (p \to q)).
\end{align*}

The rules of inference: substitution and modus ponens (MP).

In turn, $TV_→$ is an implicational fragment of the classical propositional logic. $I$, $B$, $C$, $W$, $K_1'$ is an independent axiomatization of $H_→$, and then our problem consists in finding such formula $X$ the addition of which to $I$, $B$, $C$, $W$, $K_1'$ gives an independent axiomatization of $TV_→$. The problem has the following decision:

**Theorem 1.** $I$, $B$, $C$, $W$, $K_1'$, $X$ is an independent axiomatization of $TV_→$, where $X$ is $((p \to ((q \to q) \to p)) \to (((p \to q) \to q) \to ((q \to p) \to p))).$

**Proof.** The proof consists from two parts: (i) $I$, $B$, $C$, $W$, $K_1'$, $X$ is an independent set of formulae; (ii) logic $IBCWK_1'X$ is $TV_→$. 

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(i). Matrix 1 [1, p. 85]

\[
\begin{array}{c|ccc}
\to & 0 & 1 & 2 \\
\hline
0 & 2 & 2 & 2 \\
1 & 0 & 0 & 2 & \text{verifies} & \text{falsifies} \\
\ast2 & 0 & 0 & 2 & \text{B, C, W, K'}_1, X & \text{I}(p = 1)
\end{array}
\]

Matrix 2 [1, p. 85]

\[
\begin{array}{c|cccc}
\to & 0 & 1 & 2 & 3 \\
\hline
0 & 3 & 3 & 3 & 3 \\
1 & 0 & 2 & 0 & 3 \\
\ast2 & 0 & 0 & 3 & 3 & \text{verifies} & \text{falsifies} \\
\ast3 & 0 & 0 & 0 & 3 & \text{I, B, W, K'}_1, X & \text{C}(p = 2, q = 1, r = 1)
\end{array}
\]

Matrix 3 (three–valued Łukasiewicz’s implication)

\[
\begin{array}{c|ccc}
\to & 0 & 1 & 2 \\
\hline
0 & 2 & 2 & 2 \\
1 & 1 & 2 & 2 & \text{verifies} & \text{falsifies} \\
\ast2 & 0 & 1 & 2 & \text{I, B, C, K'}_1, X & \text{W}(p = 1, q = 0)
\end{array}
\]

Matrix 4 (three–valued Sobociński’s implication)

\[
\begin{array}{c|ccc}
\to & 0 & 1 & 2 \\
\hline
0 & 2 & 2 & 2 \\
\ast1 & 0 & 1 & 2 & \text{verifies} & \text{falsifies} \\
\ast2 & 0 & 0 & 2 & \text{I, B, C, W, X} & \text{K'}_1(p = 1, q = 1, r = 2)
\end{array}
\]

Matrix 5 (three–valued Heyting’s implication)

\[
\begin{array}{c|ccc}
\to & 0 & 1 & 2 \\
\hline
0 & 2 & 2 & 2 \\
1 & 0 & 2 & 2 & \text{verifies} & \text{falsifies} \\
\ast2 & 0 & 1 & 2 & \text{I, B, C, W, K'}_1 & \text{X}(p = 2, q = 1, r = 0)
\end{array}
\]

Some difficulties arise with respect to axiom B. Let us note that we can replace equivalently the axiom C in \text{IBCWK'}_1X by shorter one, namely \text{I}:

\[p \to ((p \to q) \to q).\]

It follows from the fact that \text{IBC}\equiv \text{IBI'} (see, for example, [9, p. 178]). It means that we have only one axiom, which contains eleven letters, i.e.
axiom B. Then in virtue of Jaśkowski's result [4] (see also the review of this paper by Church [2]) axiom B is independent from I, C, W, K'1, X.

(ii). In virtue of Tarski–Bernays theorem [7, p. 145] TV→ is axiomatizable by formulae B', K and P, where

B'. \((p \rightarrow q) \rightarrow ((q \rightarrow r) \rightarrow (p \rightarrow r))\)
K. \(p \rightarrow (q \rightarrow p)\)
P. \(((p \rightarrow q) \rightarrow p) \rightarrow p\).

Formulæ B' and K are theorems of H→. So we must prove the Pearce's law, i.e. formula P:

1. K, 2.W, 3.X.
4. \(p \rightarrow ((q \rightarrow q) \rightarrow p) - 1 q/q \rightarrow q.\)
5. \(((p \rightarrow q) \rightarrow q) \rightarrow ((q \rightarrow p) \rightarrow p) - \text{MP 3,4.}\)
6. \(((p \rightarrow (p \rightarrow q)) \rightarrow (p \rightarrow q)) \rightarrow (((p \rightarrow q) \rightarrow p) \rightarrow p) - 5 q/p \rightarrow q.\)
7. \(((p \rightarrow q) \rightarrow p) \rightarrow p - \text{MP 6,2.}\)

Thus, Theorem 1 is proved.

**Theorem 2.** I, B, C1, W, K'1, X is an independent of axiomaticization an implicational fragment of modal logic S5, where C1 is a substitutional instance of axiom C, i.e.

\((p \rightarrow ((q \rightarrow r) \rightarrow s)) \rightarrow ((q \rightarrow r) \rightarrow (p \rightarrow s)).\)

**Proof.** The point (i) can be proved by the same matrices as in Theorem 1. (ii). I, B, C1, W, K'1 is an independent axiomatization of S4→. It follows from the results of Mendez [8]. Using a model for S5, suggested in [11], we can show that formula X is a theorem of S5→. It follows from [10] that Pearce's weak law P1

\(((p \rightarrow q) \rightarrow r) \rightarrow (p \rightarrow q) \rightarrow (p \rightarrow q)\)

is a characteristic axiom for S5→. P1 is provable by analogy with P in Theorem 1.

**Theorem 3.** I, B, C, W, K'1, X, N is an independent axiomaticization of classical propositional logic TV, where N is 0→p and 0 is a constant, interpreted as Falsehood.

**Proof.** The point (i) is trivial. The point (ii) follows from Wajsberg's result [12] where it is proved that the addition of axiom 0→p to an arbitrary
TV→ gives us full TV.

Now, since elements I, B, C, W, K′1, X, N are independent, i.e. they are distinct elements, let us consider the set of all subsets of \{I, B, C, W, K′1, X, N\}. In result we can construct 7-dimensional (7 axioms) cube. The elements of this cube are \(2^7(=128)\) propositional logics which are ordered by inclusion relation. We call this cube a *construction of classical propositional logic* (TV-Construction). Studing this construction is of great interest.

For example, let us consider the following projection into 3-dimensional cube, where IBCK′1 ≡ BCK:

\[
\begin{array}{c}
\text{HN} \\
\text{BCK} \\
\text{TV} \\
\text{HN} \\
\text{BCK} \\
\text{TV} \\
\text{HN} \\
\end{array}
\]

Note that BCKX is commutative BCK-logic [3] and \(L_{\aleph_0}\) is infinite-valued Lukasiewicz’s logic [7]. So in TV-Construction we have \(L_{\aleph_0}\) (!). Moreover, since logic \(L_{\aleph_0}\) is extendable to TV by addition of axiom W, then we can show that certain substitutions into axiom W generate the whole class of \(n\)-valued Lukasiewicz’s logics \((n \geq 2, n \in \mathbb{N})\) [7].

**Hypothesis.** Between two neighbourhood logics (for example, \(L_{\aleph_0}\) and TV) the infinite classes of logics are located.
Furthermore, from Theorem 2 it follows that $S_5 \rightarrow -$construction embeds into $TV \rightarrow -$construction, and due to substitution in $W, L_3 \rightarrow -$construction, for example, embeds into $TV$–construction is multi–layered.

Finally, $TV$–construction allows us to talk not only about interrelations between logics, but also about their classification. The last issue is worth special attention.

References


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