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REPRESENTATION THEOREM FOR EQUIVALENTIAL ALGEBRAS

Equivalential algebras (EA’s) were introduced by Kabziński and Wronski in [2] as an algebraic counterpart of the equivalential fragment of the intuitionistic logic (see also [1] and [4]). By an equivalential algebra we mean an algebra $U = (A, \cdot)$ with a binary operation (equivalence) satisfying the following identities for all $x, y, z \in A$ (we adopt the convention of associating to the left and ignoring the symbol of the equivalence operation):

$$xxy = y, xyzz = xz(yz), xy(xzz)(xzz) = xy.$$ We can define the unit element in $U$, by $1 := aa$ for an arbitrary $a \in A$.

The theory of EA’s is closely related to the theory of Brouwerian semilattices (see [3]), as every EA can be embedded into the $(\leftrightarrow)$–reduct of a Brouwerian semilattice, where $a \leftrightarrow b = (a \to b) \land (b \to a)$ ([2]). It is readily shown that such a reduct is an EA. In the remainder of this section we assume that an EA $U = (A, \cdot)$ is given. We say that $U$ is associative if it fulfills the associativity law. The class of all associative EA’s coincides with the class of all Boolean groups and gives an algebraic counterpart of the equivalential fragment of classical logic.

By a filter of $U$ we mean a non-empty subset $F$ of $A$ such that for all $a, b \in A$ : (i) if $a \in F$, then $abb \in F$ and (ii) if $a \in F$ and $ab \in F$, then $b \in F$. The lattice of filters of $U$ is isomorphic with the congruence lattice of $U$.

Let $x \in A$. We denote by $[x]_E$ the smallest filter containing $x$. The relation $\leq_E$ defined by $x \leq_E y$ when $y \in [x]_E$ is a partial order in $A$.

Let $x \in A - \{1\}$. We say that $x$ is irreducible if $x = ab$ implies $x = a$ or $x = b$ for all, $a, b \in A$ with $a, b \geq_E x$. By $I(\mathfrak{U})$ we denote the set of all irreducible elements of $\mathfrak{U}$. In the sequel we shall consider only the class of irreducibly generated EA’s, i.e., EA’s with the property that $\mathfrak{U} = \langle I(\mathfrak{U}) \rangle$. This class contains all finite EA’s.

The partial order $\leq_E$ restricted to $I(\mathfrak{U})$ has the following property.
Proposition 1. Let \( a, b, c \in I(\Omega) \). If \( ab \leq_E c \), then \( a \leq_E c \) or \( b \leq_E c \).

We can now introduce the equivalence relation on the set \( I(\Omega) \). Let \( a, b \in I(\Omega) \). We shall write \( a \sim b \) iff \( x <_E a \Leftrightarrow x <_E b \) for every \( x \in I(\Omega) \). Upon partitioning the set of all irreducible elements by \( \sim \), we obtain a family of associative subalgebras. Moreover, every element of \( \Omega \) can be uniquely decomposed into a finite \( \leq_E \)-antichain of irreducible elements from different equivalence classes. Namely, the following propositions hold.

Proposition 2. Let \( a \in I(\Omega) \). Then \([a]_{\sim} \cup \{1\} \) is an associative subalgebra of \( \Omega \).

Proposition 3. For every \( a \in A \), there exists a unique finite \( \leq_E \)-antichain \( \{a_1, \ldots, a_n\} \subset I(\Omega) \) such that \( a = a_1 \cdots a_n \) and \( a_i \not\sim a_j \) for all \( i, j \in \{1, \ldots, n\}, i \neq j \).

Corollary. There is a one-to-one correspondence between \( A \) and the set \( \{C \subset I(\Omega) : C \text{ is a finite } \leq_E \text{-antichain such that } x \not\sim y \text{ for all } x, y \in C\} \).

We can now try to abstract the representations of EA’s from this concrete situation.

Let \((P, \leq)\) be a poset, \( a, b \in P \). We shall write \( a \sim b \) if \( x < a \Leftrightarrow x < b \) for every \( x \in P \). Clearly, \( \sim \) is an equivalence relation on \( P \). We shall denote by \((\overline{P}, \leq)\) the extension of the poset \((P, \leq)\) with \( \overline{P} := P \cup \{1\} \), \( 1 \notin P \) and \( 1 = max(\overline{P}, \leq) \). If \( U \subset P \), we put \( \overline{U} := B \cup \{1\} \). Let \( "\cdot" \) be a partial binary operation on \( \overline{P} \) such that \( \text{dom}(\cdot) = \{U \times \overline{U} : U \in P/\sim\} \). We say that the triple \( P := (\overline{P}, \leq, \cdot) \) is an equivalential frame if:

(i) \((\overline{U}, |\overline{P}|_{\overline{U}})\) is a Boolean group with unit \( 1 = max(\overline{P}, \leq) \) for each \( U \in P/\sim \);

(ii) \( x \cdot y < z \) implies that \( x < z \) or \( y < z \) for all \( x, y, z \in P \) such that \( x \sim y \).

Let us note that the triple \( P(\Omega) := (\overline{I(\Omega)}, \leq_E, \cdot) \) fulfills the above def-
inition, where "." denotes the appropriate restriction of the equivalence operation of \( \mathfrak{A} \). Conversely, let \( \mathcal{P} := (\mathcal{P}, \leq, \cdot) \) be an equivalential frame. We index the quotient set \( \mathcal{P}/ \sim \) by \( \mathcal{P}/ \sim = \{ \mathcal{U}_j \}_{j \in I} \). Let

\[
A(\mathcal{P}) := \{ a \in \prod_{j \in I} \mathcal{U}_j : \{ a_j : j \in I \} - \{ 1 \} \text{ is a finite } \leq\text{-antichain } \}.
\]

Now we introduce the binary operation on \( A(\mathcal{P}) \). We perform Boolean group operation on each coordinate separately, but then take minimal elements only, remembering that the result has to be an antichain. Namely,

\[
(a \cdot b)_i := \begin{cases} 
  a_i \cdot b_i & \text{if } a_i \cdot b_i \in M \{ a_j \cdot b_j : j \in I \} \\
  1 & \text{otherwise}
\end{cases}
\]

where \( a, b \in A(\mathcal{P}), i \in I \) and \( M(C) \) denotes the set of all minimal elements of \( C - \{ 1 \} \) for \( C \subset \mathcal{P} \).

**Proposition 4.** \( (A(\mathcal{P}), \cdot) \) is an EA.

Let us now state our main result.

**Representation Theorem.** There is one-to-one correspondence (to within an isomorphism) between the class of all irreducibly generated EA’s and the class of all equivalential frames.

This correspondence is given by two mutually inverse (to within an isomorphism) maps: \( \mathfrak{A} \mapsto (I(\mathfrak{A}, \leq_E, \cdot)) \) and \( \mathcal{P} \mapsto (A(\mathcal{P}, \cdot)) \).

**Example.** Let us consider the equivalential algebra \( \mathfrak{A} \) being the \( (\leftrightarrow)\)-reduct of the lattice presented in Fig. 1.

![Figure 1](image_url)
Then $I(\Omega) = \{v, w, x, y, z\}$ and $[v]_\sim = \{v\}$, $[w]_\sim = \{w\}$, $[x]_\sim = [y]_\sim = [z]_\sim = \{x, y, z\}$. The equivalential frame representation of $\Omega$ is presented in Fig. 2.

![Figure 2](image-url)

Theorem 1. Let $(I, \leq)$ be a poset, let $\{V_i\}_{i \in I}$ be a family of nontrivial Boolean groups with the common unit. For every $i \in I$, let $\{A_{ij}\}_{j \in I}$ be a family of subgroups of $V_i$ such that:

(i) $i < j$ if and only if $A_{ij} \neq V_i$;
(ii) if $i j$, then there exists $k \in I$ such that $A_{ik} \neq A_{jk}$;
(iii) if $i < j < k$, then $A_{jk} \subseteq A_{ij}$.

Let us consider the set $P = \bigsqcup_{i \in I} V_i - \{1\}$ ordered by the condition that $x < y$ if and only if $x \not\in A_{ij}$, where $x \in V_i - \{1\}$ and $y \in V_j - \{1\}$, and endowed with the natural partial Boolean group operation. Then $P = (P, \leq, \cdot)$ is an equivalential frame.

Note that there is also a one–to–one correspondence between equivalential frames and structures fulfilling the hypotheses of Theorem 1.

On applying the representation theorem, we can prove the following new results:

Theorem 2. (i) every homorphic image of an irreducibly generated equivalential algebra $\Omega$ is isomorphic to a subalgebra of $\Omega$;
(ii) two finite Brouwerian semilattices with isomorphic equivalential reducts are isomorphic.
The following generalization of (ii) remains an open problem: Let $B_1$ and $B_2$ be two finite Brouwerian semilattices such that the $(\leftrightarrow)$–reduct of $B_1$ can be embedded into the $(\leftrightarrow)$–reduct of $B_2$. Is it true that $B_1 \in S(B_2)$?

Proofs of the theorems stated in the present announcement will appear in a forthcoming paper.

References


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