COMPLEXITY OF EQUATIONAL THEORY OF RELATIONAL ALGEBRAS WITH PROJECTION ELEMENTS

Abstract

In connection with a problem of L. Henkin and J.D. Monk we show that the variety generated by TPA’s – relation algebras (RA’s) expanded with concrete set theoretical projection functions – and the first–order theory of the class TPA are not axiomatizable by any decidable set of axioms. Indeed, we show that Eq(TPA) – all equations valid in TPA – is exactly on the \( \Pi^1_1 \) level. The same applies if we replace TPA’s with the expansions of RA’s suggested by P.A.S. Veloso and A.M. Haieberer in [17] as a candidate for finitary algebraization of first–order logic. Finally, we introduce TPA\(^-\), the RA–reducts of TPA, and prove that Eq(TPA\(^-\)) is recursively enumerable, but not finitely axiomatizable.

1. Introduction

The class TPA of true pairing algebras was introduced and investigated, e.g., by Maddux in ([8], [10] Definitions 17–20), and in different form, independently, by Veloso and Haieberer in [17], [18]. TPA’s in a different form are also discussed in [15], item 4.1 (iv)) on p.96, beginning with line 15. (The idea of adding the projection functions \( p \) and \( q \) as constant symbols to the equational language of relation algebras is also in ([15] pp.251–254); we mention this because this idea concerning \( p \) and \( q \) is essential in the definition of TPA’s.) The way TPA was used in the quoted works provides

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motivation to study the equational theory $\text{Eq}(\text{TPA})$ of TPA. In particular, it seems to be relevant to these works to ask whether $\text{Eq}(\text{TPA})$ can be axiomatized in some way. (In §3, we will return to discussing why and how the quoted papers motivate this question. We will also try to interpret our results in that context there.) We will investigate this, and a few related questions in §2 below.

2. Results and definitions

Given a class $K$ of algebras of the same similarity type, we denote the classes of homomorphic images, subalgebras and isomorphic copies of direct products of members of $K$ by $HK$, $SK$ and $PK$, respectively. For any set $V, \mathcal{P}(V)$ denotes the powerset of $V$. $B(V)$ denotes the Boolean set algebra whose universe is $\mathcal{P}(V)$. That is, $B(V) = \langle \mathcal{P}(V), \cup, \cap, \setminus, V, \emptyset \rangle$.

**Definition 1.** The algebra $A$ is called a *pairing (relation) algebra* (a PA) iff

$$A \subseteq \langle B(U \times U), \circ, -1, \text{Id}, p, q \rangle$$

for some set $U$, where the operations of $A$ are the following. For any $R, S \subseteq U \times U$, $R \circ S$ is the usual relational composition, $R^{-1}$ is the usual inverse of $R$, while $\text{Id}, p, q$ are constants defined below.

$$\text{Id} \overset{\text{def}}{=} \{ \langle a, a \rangle : a \in U \},$$

$$p \overset{\text{def}}{=} \{ \langle \langle a, b \rangle, a \rangle : a \in U \text{ and } \langle a, b \rangle \in U \},$$

$$q \overset{\text{def}}{=} \{ \langle \langle a, b \rangle, b \rangle : b \in U \text{ and } \langle a, b \rangle \in U \}.$$  

We note that $U$ need not contain any ordered pair like $(a, b)$, and then $p$ and $q$ will be empty. Or, e.g., if $U = \{ (0, 1), 1, 2, 3 \}$, then $p = \emptyset$, while $|q| = 1$. 

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Definition 2. ([8], [9], [10] §4, Def.20):

(i) By a simple true pairing algebra (a Si(TPA)) we understand a PA in which the equation \( p^{-1} \circ q = 1 \) is valid. In symbols,

\[
\text{Si}(\text{TPA}) = \text{PA} \cap \text{Mod}(p^{-1} \circ q = 1).
\]

(ii) TPA is the class of subalgebras of direct products of Si(TPA)’s, up to isomorphism. In symbols,

\[
\text{TPA} = \text{SPSi}(\text{TPA}).
\]

\[\square\]

A remarkable difference between PA’s and TPA’s is that while every nontrivial TPA is infinite, there are many finite PA’s. Moreover, every relation set algebra in the sense of ([7], Def.5.3.2) is a subreduct of some PA (the operations to be omitted are the constant symbols \( p \) and \( q \)). Further, the class of relational set algebras and that of the \( p, q \)-free reduct of PA’s coincide, up to isomorphism.

Proposition. The classes TPA and PA are not closed under taking ultraproducts, hence they are not even pseudo axiomatizable classes (pseudo axiomatizable is called “pseudo elementary in the broader sense” in [6], and in the handbook [2]). \[\square\]

Despite of the above proposition, the equational theory or the first-order theory of TPA could, in principle, be reasonably simple; and then one could apply (when possible) this theory instead of TPA itself. Theorem 1 below says that this is not the case. Even so, since TPA is an important class of algebras, one would like to know exactly how “hard” the theories of TPA are. This also will be answered by Theorem 1.

Notation. For a class K of algebras, Th(K) and Eq(K) denote the first-order theory and the equational theory of K, respectively.
In the formulation of Theorem 1 below, we use the terminology of recursion theory for classifying the “degrees of uncomputability” of non-computable functions. The just quoted “degrees” or complexity classes are denoted by $\Pi^k_n$ and $\Sigma^k_n (n, k \in \omega)$, where if $k > 0$, then the set in question is so far from being computable that it is called non-arithmetical, cf. e.g., ([2], §C.1.10). ($\Sigma^1_0$ means recursively enumerable. If a set is $\Sigma^2_0$, then neither the set nor its complement is recursively enumerable. And so on; climbing higher and higher in the hierarchy we get less and less “describable” sets. And $\Pi^1_1$ is infinitely higher in this hierarchy than $\Sigma^2_0$ is.)

We note that Theorem 1 implies that $\text{Eq}(\text{TPA})$ is not describable in the set theory without Axiom of Infinity.

**Theorem 1.** The set $\text{Eq}(\text{TPA})$ of all equations valid in TPA and the set $\text{Eq}(\text{PA})$ of all equations valid in PA are $\Pi^1_1$ complete (in the analytical hierarchy). That is, the “degree of unsolvability” or measure of undefinability of $\text{Eq}(\text{TPA})$ as well as that of $\text{Eq}(\text{PA})$ is exactly $\Pi^1_1$.

**Proof.** This is an immediate consequence of Theorems 3 and 4 below. □

**Definition 3.** Let SEBR denote the class of algebras which are called the standard models of the extended calculus of binary relations in [17], [18].

**Corollary.** $\text{Eq}(\text{SEBR})$ is $\Pi^1_1$ complete. The same applies to the first-order theory of SEBR.

**Proof.** The TPA-operations are term-definable in SEBR, hence our Theorem 1 immediately yields this corollary. □

**Theorem 2.**

(1) The variety generated by TPA is not finitely based (i.e., $\text{HSPTPA}$ is not finitely axiomatizable). Moreover, $\text{HSPTPA}$ is not axiomatizable by any decidable (or even recursively enumerable) set of axioms. The same applies to PA in place of TPA.
(2) *The first-order theory* \( \text{Th}(\text{TPA}) \) *of TPA is not finitely axiomatizable. Th(TPA) is not axiomatizable by any recursively enumerable set either. The same applies to PA.*

**Proof.** This is an immediate consequence of Theorem 1. \( \square \)

Let \( \mathbb{N} \overset{\text{def}}{=} \langle \omega, \text{pred}, +, \cdot \rangle \), where \( \omega \) denotes the set of natural numbers, \( \text{pred} \) is the predecessor function \( (\text{pred}(0) = 0) \), and \( +, \cdot \) stand for addition and multiplication, respectively.

**Theorem 3.** Let \( \psi \) be a \( \Pi^1_1 \) sentence of arithmetic. Then there is an equation \( e_\psi \) in the language of TPA (effectively calculable from \( \psi \)) such that

\[
\mathbb{N} \models \psi \iff \text{TPA} \models e_\psi.
\]

\( \square \)

The following theorem states that Eq(TPA) is at most on the \( \Pi^1_1 \) level, and so, by Theorem 3, it is exactly on the \( \Pi^1_1 \) level.

**Theorem 4.** Let \( e \) be an equation in the language of TPA’s. Then there is a \( \Pi^1_1 \) sentence \( \psi_e \) (effectively calculable from \( e \)) such that

\[
\text{TPA} \models e \iff \mathbb{N} \models \psi_e.
\]

\( \square \)

Let TPA\(^-\) be obtained from TPA by omitting the constant symbols \( p, q \) naming the projection functions. That is, TPA\(^-\) is that reduct of TPA which belongs to the similarity class of relation algebras (RA’s) in the sense of, e.g., ([7], §5.3). Then Eq(TPA\(^-\)) is recursively enumerable but is still not finitely axiomatizable.
**Theorem 5.** Eq(TPA⁻) is recursively enumerable. □

**Theorem 6.** Eq(TPA⁻) is not finitely axiomatizable. □

**Problem.** The concept of finiteness is somehow present in the definitions of both TPA and the algebras in [17] (in the latter explicitly, in the former via the Axiom of Foundation of set theory). One could think that this is the reason for $\Pi_1^1$ hardness of the theories of these classes. Therefore one could conjecture that Craig’s finite-sequence version of cylindric algebras (or of algebras of first-order logic) would behave similarly. That this is not the case is demonstrated, e.g., by the Completeness Theorem in ([4], §5). Since Craig has been introducing new kinds of finite-sequence algebras (e.g., at the Algebraic Logic Subsemester at Banach Center Warsaw, Sept.15–Oct.30 1991), we would like to know if one of these exhibits the highly non-axiomatizability feature of TPA’s. Here ([12], pp.105–114) and Proposition on p.193 in [3] might be relevant.)

### 3. Motivation coming from logic

The textbook [7] devotes §4.3 and a large part of §5.6 to explaining the connection between logics and their potential algebraic counterparts. According to this general theory in [7], RCA$_\omega$ and RQPA$_\omega$ are adequate algebraic counterparts of first-order logic, $L_{\omega\omega}$, with and without equality, respectively. It is exactly in this role in which SEBR (cf. Definition 3 and Corollary above) is suggested as a possible, improved substitute for RCA$_\omega$ and RQPA$_\omega$ in ([17] §§6, 7). The fact that RCA$_\omega$ is a recursively axiomatizable variety is important in the above mentioned connection between logics and algebras. Moreover, it is considered one of the central problems to find simpler axiomatization of Eq(RCA$_\omega$), cf. ([7] Problem 4.1), and/or to change RCA$_\omega$ such that it would become finitely axiomatizable and still would remain the natural algebralization of some formulation of the first-order logic (equivalent with the original one), cf. ([5], Problem 1, [11], [13] and [14]. These observations lead to the question whether Eq(SEBR) is finitely or, at least, recursively axiomatizable. The answer to this question...
(Corollary above) shows that, at least in some non-negligible aspects, SEBR does not seem to be suitable for replacing RCA_ω or RQPA_ω. (In passing, we note that the virtue that SEBR has only finitely many operation symbols was already achieved by Copeland algebras, Cpa’s, cf. ([7] p.264 item 2), without paying such a price in complexity (Eq(Cpa) is not as complex as \(\Pi^1_1\)). Actually, in [13] there is a variant K of these algebras such that Eq(K) is finitely axiomatizable.)

Another motivation for the investigations in §2 is the following. Let \(\text{Fm}(L_\omega)\) and \(\text{Mod}(L_\omega)\) be the classes of all formulas and all models of \(L_\omega\), respectively. [10] defines two functions \(\syn : \text{Fm}(L_\omega) \to (\text{a set of equations})\) and \(\sem : \text{Mod}(L_\omega) \to \text{TPA}\), giving in this way a translation both of syntax and semantics of \(L_\omega\) to algebra (\(\syn\) is \(Tr\) of Definition 17, and \(\sem\) is in Theorem 21 therein). Further, [10] defines a finitely axiomatizable variety, \(V\), such that, for every \(\varphi \in \text{Fm}(L_\omega)\), \(\vdash \varphi \iff \lor \models \syn(\varphi)\). So, in some sense, \(\lor\) could be considered as an algebraic counterpart of proof theory. Similarly, TPA (with \(\sem\)) can be considered as an algebraic counterpart of model theory because \(M \models \varphi \iff \sem(M) = \syn(\varphi)\), for all \(\varphi \in \text{Fm}(L_\omega)\) and \(M \in \text{Mod}(L_\omega)\). Studying these constructions it is natural to ask how far this “algebraic proof theory” \(\lor\) is from this “algebraic model theory” TPA. (We note that \(\lor \supset TPA \supset \{\sem(M) : M \in \text{Mod}(L_\omega)\}\).) According to our Theorem 1, the measure of the distance is exactly \(\Pi^1_1\).

There is a further logical application as follows. Consider Venema’s two dimensional modal logic with modalities rendering \(\varphi \circ \psi\), \(\varphi^{-1}\), Id to be formulas whenever \(\varphi, \psi\) are formulas, cf. [16]. A Kripke model of this logic is a pair \(\langle U \times U, \text{val} \rangle\) for some set \(U\), where \(\text{val}\) maps the propositional variables into \(\mathcal{P}(U \times U)\), and, for any \(a, b \in U\),

\[
\langle a, b \rangle \models \varphi \circ \psi \iff (\exists c \in U) \langle a, c \rangle \models \varphi \land \langle c, b \rangle \models \psi,
\]

\[
\langle a, b \rangle \models \text{Id} \iff a = b \quad \text{and} \quad \langle a, b \rangle \models \varphi^{-1} \iff \langle b, a \rangle \models \varphi.
\]

Now, let us add two constant modalities \(P\) and \(Q\) with the following semantics:

\[
\langle a, b \rangle \models P \iff (\exists c \in U) a = \langle b, c \rangle
\]

\[
\langle a, b \rangle \models Q \iff (\exists c \in U) a = \langle c, b \rangle.
\]
Now, the validity problem of this modal logic, where the set of modalities is \( \{\circ, -1, \text{Id}, P, Q\} \), is \( \Pi^1_1 \) hard, an immediate consequence of Theorem 1.

References


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