Abstract

We prove that the existence of consistent choices for dense relations, in short $\mathcal{F}_{\text{fin}}$, is tantamount in $ZF$ to the existence of such choices for $n$–element sets with $n \geq 3$. Our interest in $\mathcal{F}_{\text{fin}}$ stems from [2] where it has been shown that $\mathcal{F}_{\text{fin}}$ is equivalent to $CT_{\text{fin}}$ – the Compactness Theorem for locally finite sets of propositional sentences. The condition $CT_{\text{fin}}$, in turn, was suggested by Cowen [1] in connection with the $P = NP$ – problem.

The question whether $\mathcal{F}_{\text{fin}}$ is equivalent to $AC_{\text{fin}}$, as well as the equivalence of $\mathcal{F}_{2}^{\text{fin}}$ and $AC_2$ are still open.

A binary relation $\mathcal{R}$ on a set $\mathcal{X}$ is said to be dense iff for every $a \in \mathcal{X}$ \{b $\in X : \neg a\mathcal{R}b\}$ is finite;

let $\mathcal{A}$ be a family of sets and $\mathcal{R}$ a binary symmetric relation on $\bigcup \mathcal{A}$. A choice set $S$ on $\mathcal{A}$ is $\mathcal{R}$–consistent iff $a\mathcal{R}b$ for every $a \neq b$, $a, b \in S$.

The existence of consistent choices, and its variants, are formulated as follows:

$\mathcal{F}$ – for every nonempty family $\mathcal{A}$ of pairwise disjoint, finite and nonempty sets and a binary symmetric relation $\mathcal{R}$ on $\bigcup \mathcal{A}$ if every finite subfamily of $\mathcal{A}$ has an $\mathcal{R}$–consistent choice, then the whole family $\mathcal{A}$ has such a choice set;

$\mathcal{F}_{\text{fin}}$ – as above with additional assumption that $\mathcal{R}$ is dense;

$\mathcal{F}_{n}^{\text{fin}}$ – as above for families of exactly $n$–element sets;

$\mathcal{F}_{\leq n}^{\text{fin}}$ – as in $\mathcal{F}_{n}$ for families of at most $n$–element sets.
Facts.

**Theorem 1.** \( \mathcal{F}_{n+1}^{fin} \rightarrow \mathcal{F}_n^{fin} \) for every \( n \geq 2 \).

**Proof.** Let every finite subfamily \( \mathcal{A}_0 \) of \( \mathcal{A} \) has an \( \mathcal{R} \)-consistent choice. With every \( A \in \mathcal{A} \) we correlate two distinct new elements, say \( *_A^1 \) and \( *_A^2 \). Then we put \( A_1^* := A \cup \{*_A^1\} \), \( A_2^* := A \cup \{*_A^2\} \), where \( A_1^*, A_2^* \) are two disjoint “copies” of \( A \) e.g. \( A_1^1 := \{(a,1) : a \in A\} \), \( A_2^2 := \{(a,2) : a \in A\} \). Moreover let \( \mathcal{A}^* := \{A_i^* : A \in \mathcal{A}, i = 1,2\}, \ A^* := A_1^* \cup A_2^* \) and \( \pi((a,i)) = a \) for \( a \in A, i = 1,2 \). It should be noticed that \( \mathcal{A}^* \) is a family of \( n+1 \)-element sets.

\[
\begin{array}{cccc}
A & : & a & b & \ldots & z \\
\hline
A_1^* & : & a^1 & b^1 & \ldots & z^1 & *_A^1 \\
| & | & | & | & | & | \\
A_2^* & : & a^2 & b^2 & \ldots & z^2 & *_A^2 \\
\hline
\end{array}
\]

A binary symmetric relation \( \mathcal{R}^* \) on \( \bigcup \mathcal{A}^* \) is defined as follows (see above diagram):

1. If \( x \in A^*, y \in B^*, A \neq B, A,B \in \mathcal{A} \), then
   - if \( x \) or \( y \) is an “asterisk”, then \( x \mathcal{R}^* y \);
   - otherwise \( x \mathcal{R}^* y \) iff \( \pi(x) \mathcal{R} \pi(y) \),

2. If \( x,y \in A^*, A \in \mathcal{A} \), then
   - if one of \( x \) or \( y \) is an “asterisk” then they are not in \( \mathcal{R} \);
   - otherwise \( x \mathcal{R}^* y \) iff \( \pi(x) = \pi(y) \).
Of course $\mathcal{R}$ is dense, hence by $\mathcal{F}_{n+1}^{\text{fin}}$, then exists an $\mathcal{R}$–consistent choice $S^*$ of $A^*$. Let us notice that for no $A \in \mathcal{A}$, and no $j = i, 2 \ast i \in S^*$. Let $A \in \mathcal{A}$. If $S^* \cap A^1 = \{a_1\}$ and $S^* \cap A^2 = \{a_2\}$ then $a_1 \mathcal{R} a_2$ which shows, by the definition of $\mathcal{R}$, that $a_1$ and $a_2$ are the copies of the same element of $A$. Thus the set $S := \{\pi(x) : x \in S^*\}$ is a choice on $\mathcal{A}$. It is not difficult to notice that $S$ is $\mathcal{R}$–consistent. □

**Corollary 1.** $\mathcal{F}_{n}^{\text{fin}} \rightarrow \mathcal{F}_{n+1}^{\text{fin}} \rightarrow \mathcal{F}_{n}^{\text{fin}} \rightarrow \mathcal{F}_{3}^{\text{fin}} \rightarrow \mathcal{F}_{2}^{\text{fin}}$.

**Corollary 2.** $\mathcal{F}_{n}^{\text{fin}} \rightarrow \bigwedge_{k \leq n} AC_k$.

**Proof.** One can easily show that $\mathcal{F}_{k}^{\text{fin}} \rightarrow AC_k$, for every $k \geq 2$.

**Theorem 2.** $\mathcal{F}_{3}^{\text{fin}} \rightarrow \mathcal{F}_{\leq 3}^{\text{fin}}$.

**Proof.** Let $\mathcal{A}$ be a family of at most three–element, nonempty and pairwise disjoint sets and $\mathcal{R}$ be a binary symmetric relation on $\bigcup \mathcal{A}$ such that every finite subfamily $\mathcal{A}_0$ of $\mathcal{A}$ has an $\mathcal{R}$–consistent choice. For any $A \in \mathcal{A}$ we define three–element set $A^*$ in a following way:

1. If $A$ is three–element, we just put $A^* := A$;
2. If $A$ is two–element, we choose $a_A \in A$ (by $AC_2$–comp. Cor. 2), and take $A^* := A \cup \{a_A^*\}$, where $a_A^*$ is a fresh “copy” of $a_A$;
3. If $A$ has exactly one–element $a$, we choose its two “copies” $a^*$ and $a^{**}$, and take $A^* := A \cup \{a^*, a^{**}\}$.

Then we take $\mathcal{A}^* := \{A^* : A \in \mathcal{A}\}$ and we define a binary relation $\mathcal{R}^*$ on $\bigcup \mathcal{A}^*$ in the following way:

$$x \mathcal{R}^* y \text{ iff } \pi(x) \mathcal{R} \pi(y), \text{ for } x, y \in \mathcal{A}^*.$$
where 

$$\pi(x) := \begin{cases} a_A & \text{if } x = a_A^*, a_A^{**} \\ x & \text{otherwise} \end{cases}$$

Now the number of those elements $y \in \bigcup \mathcal{A}^*$ such that $\neg (x \mathcal{R}^* y)$, is not greater than twice the number of those $a \in \bigcup \mathcal{A}$ for which $\neg (\pi(x) \mathcal{R} a)$. Thus $\mathcal{R}$ is dense.

Since every $\mathcal{R}$–consistent choice on $\mathcal{A}$ is also an $\mathcal{R}^*$–consistent choice on $\mathcal{A}^*$, we get an $\mathcal{R}^*$–consistent choice $\mathcal{S}$ on the family $\mathcal{A}^*$. Then we easily see that $\{\pi(x) : x \in \mathcal{S}\}$ is an $\mathcal{R}$–consistent choice on $\mathcal{A}$. □

As it is known (see [2]) $\mathcal{F}_{f_n}$ is equivalent to some statement about propositional calculus. We consider the language $\{\neg, \land, \lor\}$ and accept standard definitions of propositional formulae.

A set $\mathcal{X}$ of propositional formulas is said to be locally satisfiable iff every finite subset $\mathcal{X}_0$ of $\mathcal{X}$ is satisfiable; $\mathcal{X}$ is locally finite iff every variable $p$ of $\mathcal{X}$ appears only in finite number of formulas of $\mathcal{X}$. The Compactness Theorem and its variants are the following statements:

$CT$ – every locally satisfiable set of propositional formulas is satisfiable;

$CT_{f_n}$ – every locally satisfiable and locally finite set of propositional formulas is satisfiable;

$n$–Sat $-$ every locally satisfiable set of elementary disjunctions consisting of at most $n$ literals ($a$ literal is a variable or its negation) is satisfiable;

$n$–Sat$_{f_n}$ – as above for locally finite sets of elementary disjunctions;

Theorem 3. $\mathcal{F}_{\leq 3} \rightarrow 3$–Sat$_{f_n}$.

Proof. Let $\mathcal{X}$ be locally satisfiable family of at most three–literal disjunctions in which every variable appears only finitely often. Let now for $\alpha \in \mathcal{X}$,

$$A_\alpha := \{(l, \alpha) : l \text{ is a literal in } \alpha\}$$
and let $\mathcal{A} := \{A_{\alpha} : \alpha \in X\}$.

Of course $\mathcal{A}$ is a family of non-empty, at most three-element, pairwise disjoint sets. Taking $R$ defined as follows:

$$(l_1, \alpha_1) \mathcal{R} (l_2, \alpha_2) \text{ iff } \{l_1, l_2\} \text{ is satisfiable,}$$

we easily see that $\mathcal{A}$ and $R$ satisfy premises of $F_{\leq 3}^{fin}$. Indeed, the density of $R$ follows from local finiteness of $X$. Let now $\mathcal{A}_0$ be a finite subfamily of $\mathcal{A}$. Then there is a finite $X_0 \subseteq X$ such that $\mathcal{A}_0 \subseteq \{A_{\alpha} : \alpha \in X_0\}$. Hence $X_0$ is satisfied by some valuation $v_0$. Since elements of $X_0$ are elementary disjunctions, the sets

$$\{(l, \alpha) \in A_{\alpha} : v_0(l) = 1\}, \text{ for } \alpha \in X_0$$

are nonempty. We know that for finite families AC holds, so we can choose exactly one $(l_{\alpha}, \alpha)$ from each of the above sets. Hence the set $\{(l_{\alpha}, \alpha) : \alpha \in X_0\}$ is an $R$–consistent choice on $\mathcal{A}_0$. Thus $\mathcal{A}$ has an $R$–consistent choice $S$. Putting now for any variables $p$:

$$v(p) := 1 \text{ iff } (p, \alpha) \in S \text{ for some } \alpha \in X$$

we get a valuation satisfying $X$. □

**Corollary 3.** $F_{n}^{fin} \rightarrow 3\text{-Sat}_{fin}$, for $n \geq 3$.

R. Cowen proved in [1] that $3\text{-Sat} \leftrightarrow CT$. The same argument can be used to establish $3\text{-Sat}_{fin} \rightarrow CT_{fin}$. Thus,

**Corollary 4.**

(i) $F_{n}^{fin} \leftrightarrow F_{fin}$, $n \geq 3$,

(ii) $F_{n}^{fin} \leftrightarrow CT_{fin}$, $n \geq 3$. 16
Proof. The proof follows clearly from the above considerations and from [2].

The above could be seen as a variant of \( F_n \leftrightarrow F, n \geq 3 \), proved by Levy [3] and Mycielski [5]. Let us, however, note that the arguments are quite different.

As an immediate corollary of the above considerations we also obtain

\[
AC_n \vdash F_n^{\text{fin}}, \quad \text{for} \quad n \geq 3.
\]

In case of

\[
AC_2 \rightarrow F_2^{\text{fin}} \quad \text{and} \quad AC_{\text{fin}} \rightarrow F_{\text{fin}}
\]

the problem is still open.

Remark 1. Levy proves in [3] that \( F_2 \vdash AC_3 \). Hence \( F_2^{\text{fin}} \) is not equivalent to \( F_{\text{fin}} \). Besides, using similar method as Levy, we can demonstrate that \( F_2 \vdash AC_k \) for every \( k \neq 2,4 \). This shows \( F_2^{\text{fin}} \vdash AC_k \) for \( k \neq 2,4 \).

Warning. Implication \( AC_{\text{fin}} \rightarrow F_{\text{fin}} \) cannot be falsified with the popular technic of “permutation models”. It is known that \( AC_{\text{fin}} \rightarrow AC_{\text{count}} \) is true in any permutation model of ZFA (see [4], pp. 114, ex 11) and \( AC_{\text{count}} \rightarrow F_{\text{fin}} \) is proved in [2].

References


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