A RELATION BETWEEN NAMESAKES IN MODAL LOGIC

Abstract

The closure under a certain rule of the modal system usually called “K4” is shown to be the modal system called “K4” by Sobociński.

A modal system S provides double cancellation iff for all sentences A and B, if $\vdash S LA \equiv LB$ and $\vdash S MA \equiv MB$ then $\vdash S A \equiv B$. It is of course not required that $\vdash S ((LA \equiv LB) \& (MA \equiv MB)) \supset (A \equiv B)$. If S is any modal system, there is a smallest normal extension $S^+$ of S providing double cancellation (terminology as in [1]). For reasons given in [3], the double cancellation rule may be of some philosophical interest. This note extends the results of [3] and [4] on double cancellation in normal systems.

The name “K4” is ambiguous. It is now customarily used for the smallest normal system containing the schema $LA \supset LLA$; this practice is followed here. Sobociński used the same name for the smallest system containing $KT4 (= S4)$ and the schemata $A \supset (MLA \supset LA)$ and $MLA \equiv LMA$ ([2]); this normal system is here referred to as $K_{4Sob}$. The relation to be established between these namesakes is that $K4^+ = K_{4Sob}$; it will also be shown that $K_{4Sob}$ is the only normal extension of K4 providing double cancellation in which there is not modal collapse. This extends the results of [3], in which it was shown that $KD4^+ = K_{4Sob}$, where $KD4!$ is the smallest normal system containing the formula $MT$ (T being a constant tautology) and the schema $LLA \equiv LA$, and [4], in which it is shown that $KW^+$ is inconsistent, where KW is “provability logic”, the smallest normal system containing $L(LA \supset A) \supset LA$, an extension of K4.

Lemma A. $\vdash_{K4^+} (A \& M(A \& LA)) \supset LA$. 

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Proof. For any formula $A$, the following are theorems of $K_{4^+}$ (where on the right a subsystem of $K_{4^+}$ is given in any normal extension of which the reasoning can be carried out; e.g. “$K$” means that the reasoning can be carried out in any normal system):

1. $(A \& (L(\neg A \lor M(\neg A \& L(\neg A)))) \supset A) \quad \text{PC}$
2. $L(A \& (L(\neg A \lor M(\neg A \& L(\neg A)))) \supset LA) \quad 1, K$
3. $M(A \& (L(\neg A \lor M(\neg A \& L(\neg A)))) \supset MA) \quad 1, K$
4. $LA \supset (L(\neg A \lor M(\neg A \& L(\neg A)))) \quad \text{PC}$
5. $LLA \supset L(L(\neg A \lor M(\neg A \& L(\neg A)))) \quad 4, K$
6. $LA \supset LLA \quad K_{4^+}$
7. $LA \supset L(L(\neg A \lor M(\neg A \& L(\neg A)))) \quad 5, 6, PC$
8. $LA \supset L(A \& (L(\neg A \lor M(\neg A \& L(\neg A)))) \quad 7, K$
9. $LA \equiv L(A \& (L(\neg A \lor M(\neg A \& L(\neg A)))) \quad 2, 8, PC$
10. $A \supset (M(A \& L(\neg A \lor M(A \& L(\neg A)))) \quad PC$
11. $MA \supset (MM(A \& L(\neg A \lor M(A \& L(\neg A)))) \quad 10, K$
12. $MM(A \& L(\neg A \lor M(A \& L(\neg A)))) \supset M(A \& L(\neg A \lor M(A \& L(\neg A)))) \quad K_{4^+}$
13. $(A \& L(\neg A \lor M(A \& L(\neg A)))) \supset (A \& L(\neg A \lor M(A \& L(\neg A)))) \quad PC$
14. $M(A \& L(\neg A \lor M(A \& L(\neg A)))) \supset M(A \& L(\neg A \lor M(A \& L(\neg A)))) \quad 13, K$
15. $MM(A \& L(\neg A \lor M(A \& L(\neg A)))) \supset M(A \& L(\neg A \lor M(A \& L(\neg A)))) \quad 12, 14, K$
16. $MA \equiv M(M(A \& L(\neg A \lor M(A \& L(\neg A)))) \quad 11, 15, K$
17. $MA \equiv M(A \& L(\neg A \lor M(A \& L(\neg A)))) \quad 3, 16, PC$
18. $A \equiv (A \& L(\neg A \lor M(A \& L(\neg A)))) \quad 9, 17, K_{4^+}$
19. $(A \& M(A \& L(\neg A))) \supset LA \quad 18, PC$
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Lemma B. \( \vdash_{K4+} (A \& LM) \supset LA \).

Proof. The following are the theorems of \( K4+ \):

1. \( L(A \& M(\sim A \& MA)) \supset (LA \& LM(\sim A \& MA)) \)

2. \( LA \supset M(\sim A \& MA) \)

3. \( (LA \& M(\sim A \& MA)) \supset (\sim A \& M(A \& M \sim A)) \)

4. \( (LLA \& LM(\sim A \& MA)) \supset L(\sim A \& M(A \& M \sim A)) \)

5. \( (LA \& LM(\sim A \& MA)) \supset L(\sim A \& M(A \& M \sim A)) \)

6. \( L(A \& M(\sim A \& MA)) \supset L(\sim A \& M(A \& M \sim A)) \)

7. \( (A \& M(A \& M \sim A)) \supset L(A \& M(\sim A \& MA)) \)

8. \( L(A \& M(\sim A \& MA)) \equiv L(\sim A \& M(A \& M \sim A)) \)

9. \( A \& LA \supset \sim M(A \& LA) \)

10. \( (A \& M \sim A) \supset L(\sim A \lor M \sim A) \)

11. \( (A \& M \sim A) \supset LL(\sim A \lor M \sim A) \)

12. \( M(\sim A \& MA) \supset M(\sim A \& MA) \)

13. \( (A \& M(\sim A \& MA)) \supset LL(\sim A \lor M \sim A) \)

14. \( M(\sim A \& MA) \& LL(\sim A \lor M \sim A) \supset M(\sim A \& MA \& L(\sim A \lor M \sim A)) \)

15. \( (\sim A \& MA) \supset L(\sim A \lor M \sim A) \)

16. \( M(\sim A \& MA \& L(\sim A \lor M \sim A)) \supset M(\sim A \& M(A \& M \sim A)) \)

17. \( (M(\sim A \& MA) \& LL(\sim A \lor M \sim A)) \supset M(\sim A \& M(A \& M \sim A)) \)

18. \( (A \& M(\sim A \& MA)) \supset M(\sim A \& M(A \& M \sim A)) \)
Lemma C. ⊢_{K4+} LLA ≡ LA.

Proof. The following are theorems of K4+:

(1) (LA & ∼ LA) ⊢ A
(2) (LLA & L ∼ LA) ⊢ LA
(3) LLA ⊢ (MLA ∨ LA)
(4) MLLA ⊢ (MMLA ∨ MLA)
(5) MMLA ⊢ MLA
(6) MLLA ⊢ MLA
(7) LA ⊢ LLA
(8) MLA ⊢ MLLA
(9) MLLA ≡ MLA
(10) (LAV ∼ LLA) & LM(LAV ∼ LLA) ⊢ L(LAV ∼ LLA)
(11) (LAV ∼ LLA) & LLA ⊢ LA
(12) (L(LAV ∼ LLA) & LLLA) ⊢ LLA
(13) (LM(LAV ∼ LLA) & LLLA) ⊢ LLA
Lemma D. \( \vdash_{K4+} MT \).

**Proof.** The following are theorems of \( K4+ \):

(1) \( LLMT \equiv LMT \) \hspace{1cm} \text{Lemma C}
(2) \( LL \sim T \equiv L \sim T \) \hspace{1cm} \text{Lemma C}
(3) \( MMT \equiv MT \) \hspace{1cm} 2, K
Lemma E. \( K4+ = K4_{Sub} \).

Proof. It is shown in [3] that \( KD4!+ = K4_{Sub} \). By Lemmas C and D, \( KD4! \subseteq K4+ \), so \( KD4!+ \subseteq K4 ++ = K4+ \). But \( K4 \subseteq KD4! \), so \( K4+ \subseteq KD4!+ \).

Lemma F. The only normal proper extensions of \( K4_{Sub} \) are \( \text{Triv} (K \text{ plus } A \equiv LA) \) and the inconsistent system.

Proof. Let \( S \) be any normal extension of \( K4_{Sub} \). By results noted in [3], the frame of any generated submodel of the canonical model of \( K4_{Sub} \) consists of either a single reflexive point or two non-symmetrically related reflexive points, and \( K4_{Sub} \) is valid on all and only such frames. Since any non-theorem of \( S \) is falsified in a generated submodel of the canonical model of \( K4_{Sub} \) that is a model of \( S \), \( S \) has the finite model property, and so by Corollary 8.14 of [1] is characterized by a class \( C \) of frames, each of must be of one of the two mentioned kinds (else it would not be a frame for \( K4_{Sub} \), and so not for \( S \)). If \( C \) contains a two-point frame, \( S = K4_{Sub} \).
If \( C \) is non-empty but contains only one-point frames, \( S = \text{Triv} \). If \( C \) is empty, \( S \) is inconsistent.

**Theorem.** The normal extensions of \( K4 \) providing double cancellation are \( K4_{Sob}, \text{Triv} \) and the inconsistent system.

**Proof.** Any normal extension of \( K4 \) providing double cancellation is one of the three mentioned systems by Lemmas E and F. \( K4_{Sob} \) provides double cancellation by Lemma E. \( \text{Triv} \) and the inconsistent system provide it trivially.

**References**


