DECISION PROBLEM FOR LINEAR ORDERINGS IN STATIONARY LOGICS

Let $T_{LO}(aa)$ be a theory of linear orderings in the stationary logic $L(aa)$. It is well-known that $T_{LO}(aa)$ is undecidable [Se 82]. On the other hand the decision problem for the theory $T_{DLO}(aa)$ of the class of finitely determinate linear orderings in $L(aa)$ remains open. The following results contribute to a solution of this problem. A linear ordering $(A, <)$ is said to be $\omega_1$-dense iff for all $a, b \in A, a < b$:

$$|\{x : a < x < b\}| \geq \omega_1$$

$(A, <)$ is called $\omega_1$-discrete iff it does not contain a subordering which is $\omega_1$-dense. Let $L(\omega_1)$ be the smallest class of linear orderings containing 1 and closed with respect to $\alpha + \beta, \omega$- and $\omega^*$- sums, $\omega_1$- and $\omega_1^*$-sums and $\eta$-sums ($\eta$ is the order type of the rational numbers).

**Proposition 1.** [He 91]

1. Let $A$ be a linear ordering of power $\omega_1$. Then either $A$ is $\omega_1$-discrete or there is an $\omega_1$-dense linear ordering $C$ and $\omega_1$-discrete orderings $A_c, c \in C$, such that $A = \Sigma_{c \in C} A_c$

2. A linear ordering $A$ of power $\leq \omega_1$ is $\omega_1$-discrete if and only if $A \in L(\omega_1)$.

**Definition 1.**

1. A linear ordering $B$ is weakly separable if it is short (i.e. neither $\omega_1$ nor $\omega_1^*$ is embeddable in it) and every densely ordered subordering $X \subseteq B, |X| > 1$, contains a non-empty open interval which is separable.

2. Let $M_{WS}$ be the smallest class of linear orderings containing 1 and closed with respect to $\alpha + \beta, \omega$- and $\omega^*$-sums, $\eta$-sums and ordered sums over $\omega_1$-dense separable orderings.
Proposition 2. [He 91]

(1) A linear ordering $\mathcal{A}$ is weakly separable if and only if $\mathcal{A} \in M_{SW}$.
(2) Every weakly separable linear ordering is finitely determinate.
(3) The theory $Th_{aa}(LO_{WS})$ in $L(aa)$ of the class $LO_{WS}$ of all weakly separable linear orderings is decidable.

Let $M_D$ be the smallest class of orderings containing 1 and closed with respect to finite sums, $\alpha \cdot \omega, \alpha \cdot \omega^*, \alpha \cdot \omega_1, \alpha \cdot \omega_1^*$ and to the shuffling operation $sh_\eta(\Delta)$, $\Delta$ - a finite set of order types (see [La 66]).

Proposition 3. [He 91] Let $K$ be the class of all finitely determinate $\omega_1$-discrete linear orderings.

(1) $Th(M_D) = Th(K)$.
(2) $Th(K)$ is decidable.

An $\omega_1$-dense linear ordering $(A, <)$ is said to be a Specker ordering if it is short and there is no uncountable subset $X \subseteq A$ which is embeddable in the ordering of the reals. $(A, <)$ is a special Specker ordering if it is finitely determinate and satisfies the following sentence from $L(aa)$:

$aaX(\lim_s(X) = X)$ ($\lim_s(X)$ is the set of both sided limit points of $X$).

Proposition 4. [He 91]

(1) There is a sentence $\Phi \in L(aa)$ such that for every short and finitely determinate linear ordering $\mathcal{A}$ holds: $\mathcal{A}$ is a Specker ordering iff $\mathcal{A} \models \Phi$.
(2) The $L(aa)$-theory of the class of special Specker orderings is decidable.

Let $sh_\eta(\Delta)$ be the shuffling operation over the type $\eta$ and $sh_\lambda(\Delta_1, \Delta_2)$, $sh_\delta(\Delta)$ the analogous operations w.r.t. the real ordering $\lambda$ and to a fixed special Specker ordering $\delta$. Let $M$ be the smallest class of linear orderings containing 1 and closed w.r.t. $\alpha + \beta, \alpha \cdot \omega, \alpha \cdot \omega^*, \alpha \cdot \omega_1, \alpha \cdot \omega_1^*, sh_\eta(\Delta), sh_\lambda(\Delta_1, \Delta_2)(\Delta, \Delta_1, \Delta_2$ - finite sets of order types).

Proposition 5. [He 91]

(1) Every ordering in $M^*$ is finitely determinate.
(2) The $L(aa)$-theory of $M^*$ is decidable.
References


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