AN UNDECIDABLE PROBLEM FOR REGULAR EQUATIONS

Abstract
This is a continuation of our note [4]. We deal with two kinds of special equations: normal and regular (see [4]). Our aim is to point an undecidable problem on regular (normal) equations. An extended version is submitted to Algebra Universalis.

Preliminaries

Our nomenclature and notation is basically those of [4]. We consider varieties of universal algebras of a type $\tau : T \to N$, where $T$ is a set and $N$ denotes the set of all positive integers. If $A$ is an algebra and $V$ is a variety (of type $\tau$), then $E(A)$ and $E(V)$ denotes the set of all equations of type $\tau$, satisfied in $A$ and $V$, respectively. Following Plonka, an equation $p = q$ is called regular if the set of all variables occurring in $p$ and $q$ coincides. An equation is called normal, if $p$ and $q$ are the same variables or neither $p$ nor $q$ is a variable. For a variety $V$, $R(V)$ and $N(V)$ denotes the set of all regular (normal) equations satisfied in $V$, respectively. Analogously, for an algebra $A$ we define sets $R(A)$ and $N(A)$. If $E(V) = R(V)$ ($E(V) = N(V)$), for a variety $V$, then $V$ is called regular (normal). From now on we consider only varieties of a finite type $\tau$, i.e. such that $T$ is a finite set.

The aim of this paper is to present a proof of undecidability of a property ($R$) for regular equations ($N$) for normal equations). Following Markov [5] we consider finite associative systems, the elements of which are
“words” – i.e. strings of letters belonging to a finite alphabet. Each system is defined by a finite number of generating relations of the form $P \Leftrightarrow Q$, where $P$ and $Q$ are words.

A property $\mathcal{P}$ of associative systems is called invariant if every system which is isomorphic to a system possessing the property $\mathcal{P}$ itself possesses this property. Let $\mathcal{P}$ be an invariant property such that (1) there is a system $S_0$ which does not have the property $\mathcal{P}$ and is not isomorphic to a subsystem of any system having the property $\mathcal{P}$, and (2) there is a system $S_1$ which has the property $\mathcal{P}$.

The main result established in [5] (cf. [6]) is that for no property $\mathcal{P}$ satisfying conditions (1) and (2) does there exist an algorithm permitting one to decide in a finite number of steps whether an arbitrary given associative system does or does not possess the property $\mathcal{P}$. If $\mathcal{P}$ is an hereditary property, i.e. if every subsystem of a system with the property $\mathcal{P}$ always has the property $\mathcal{P}$, then the condition (1) can be simplified: it is sufficient to assume that there exists a system not having the property $\mathcal{P}$.

Properties satisfying conditions (1) and (2) are referred as Markov’s properties.

Following [2] we study some problems of equational theories by transforming them into problems in monoids. A regular equational theory $E$ is called “monadic” iff there exists a presentation of this theory such that all terms in it are built by unary functional symbols only. We apply Lemma 3.11 of [2] which states the connection between $E$-equality induced by monadic theory and the equality “$=_{M}$” in the corresponding monoid $M_E$.

Given an associative system $A$. Consider a property $(R1)$ defined as follows:

\[(R1) \quad E(A) \neq R(A).\]

It is obvious that $(R1)$ is a hereditary property and that there exist associative systems with and without this property. For example the trivial monoid (on one letter $a$ and the relation $aa = a$) possesses $(R1)$ and a free monoid (on one letter $A$ and the relation $a = a$) does not possess the property $(R1)$. Therefore the property $(R1)$ for associative systems is undecidable.
From now on, suppose that $V$ is finitely axiomatized equational theory (a variety) of a finite, unary type (i.e. $\tau(T) = \{1\}$). It is well known that regular (normal) equations are closed under Birkhoff’s rules of inferences (i) – (iv) (see [1], [3], p. 170). Therefore it is decidable if a variety $V$ is regular (i.e. $E(V) = R(V)$). Namely $V$ is regular iff all equations of an axiomatic $\Sigma$ of $V$ are regular. This explains, the definition of the property $(R)$ below:

$$(R) \quad E(V) = R(W) \neq E(W),$$

for a variety $W$ of type $\tau$.

**DEFINITION.** An equational theory $V$ is called $(R)$ theory iff $V$ possesses the property $(R)$.

**THEOREM.** The class problem for regular $(R)$ theories is undecidable.

**PROOF.** We show that $(R)$ is Markov’s property. Let $S$ be the theory defined by all regular equations of a given finite unary type $\tau$. By Theorem 4 of [7], $S$ is finitely axiomatized monadic regular theory with the property $(R)$, namely $S = R(T_{\tau})$, where $T_{\tau}$ denotes the trivial equational theory of the type $\tau$. From the other hand, let $V$ be an equational theory of the type $\tau$, defined by the equation $x = x$. Let $S$ be the monoid of terms associated with $S$ (see [2], p. 13, 14). From an observation of A. Tarski [8], that the lattice of all equational theories of a given (non-empty and non-nullary) type, has no minimal non-zero elements, it follows that $S$ is not an $(R)$ theory (we can also use Theorem 5 of [4], to prove that $S$ does not have the property $(R)$). Let $S$ be the monoid associated with $S$ and $A$ be a monoid. Then if $S$ is a submonoid of $A$, then $E(S) \supseteq E(A)$, so if $E(A) = R(B) \neq E(B)$, for a monoid $B$ then $E(S) \supseteq R(B) \supseteq E(S)$ and thus $E(S) = R(B)$ which is impossible by Theorem 5 of [4]. Therefore $S$ is not a submonoid of any monoid $A$ with the property $(R)$. We conclude that the property $(R)$ is undecidable. $\Box$

Analogously one can define and consider an undecidable property $(N)$ defined in connection with the notion of normal equation.
References


*Institute of Mathematics
Polish Academy of Sciences
ul. Kopernika 18
51–617 Wrocław, Poland*