Let $F_p$ be a set of all properly binary Boolean functions $f : 2^2 \rightarrow 2$, i.e., $f$ depends on both arguments. Of the 16 binary truth functions 10 belongs to $F_p$, namely $\lor, \land, \leftrightarrow, \leftarrow$ (reverse implication), $\uparrow$ (Sheffer function) and the duals of these. For $F \subseteq F_p$ let $\vdash F$ denote the common logic\(^1\) of the $f \in F$ in the propositional language with one binary function symbol, $\vdash F = \bigcap_{f \in F} [\vdash 2^f]$, where $2^f$ denotes the 2-element matrix $((2^f,f),1)$. A study of $\vdash F$ is useful for various purposes, e.g. for information processing, see [3]. $\vdash F$ axiomatizes the common sequential rules of the $f \in F$. It needs not to have tautologies but this is a minor point. Particularly interesting is the question how ambiguous $\vdash F$ actually is, i.e., how much information in form of additional rules needs a system of information processing dealing with $\vdash F$ in order to identify a connective $f \in F$. This clearly amounts to an analysis of the strenghtenings $\vdash \supset \vdash F$. Our main result is

**Theorem 1.** $\vdash F_p$ (and hence $\vdash F$ for each $F \subseteq F_p$) has finitely many strenghtenings only. All these are determined by finitely many finite matrices.

In other words, $\vdash F_p$ has finite degree of maximality (see [4] for basic notions). $\vdash F_p$ has a huge number of strenghtenings. Presently we only know that its number is less than $10^{15}$. However, it has precisely 36 maximal

\(^1\) A logic is here a structural consequence relation denoted by $\vdash$ or a similar symbol $\vdash$ is non-trivial if not $\alpha \vdash \beta$ for all formulas $\alpha, \beta$. We omit the improper binary truth functions from our consideration because they are less interesting and cause some additional technical problems.
(nontrivial) strengthenings, including the $\models_{2^f}$ for $f \in F_p$. The remaining 26 are $2^k$-valued, $2 \leq k \leq 5$. For $|F| \leq 4$ the maximality degree of $\models_F$ is relatively small and can be computed by hand.

Theorem 1 easily follows from Theorem 2 and the Lemma below. $SK, PK$ denote the class of submatrices and of direct products of members of a class $K$ of matrices, respectively. $t_0$ and $t_1$ denote the 1-element matrices whose element is designated and not designated, respectively. $K \neq \emptyset$ implies $t_1 \in PK$ ($t_1$ appears as the power of some $A \in K$ with the empty index set). If $K, M$ are classes of matrices or single matrices we write $M \equiv K$ ($M$ is isomorphic to $K$). Clearly $K \cup \{t_0\} \equiv K$ and $K \cup \{t_1\} \equiv K$ only if $\models_K$ has no tautologies. A matrix $A$ is trivial if either $A \equiv t_0$ or $A \equiv t_1$. Call $K$ closed if for each nontrivial $A \in SK$ there is some $M_A \subseteq K$ with $A \equiv M_A$. If $\models = \models_K$ for some closed $K$ then $K$ is said to be a closed semantics for $\models$.

**Lemma ([3]).** $\models$ has finite degree of maximality iff $\models$ has a closed semantics $M, M$ finite. If $|M| = n$ then $\models$ has maximality degree $< 2^{n+1}$.

The proof follows essentially from a well-known result of [4] which implies that $K$ is closed iff each $\models' \supseteq \models_K$ has a representation $\models' = \models_K$ for some $K' \subseteq K \cup \{t_0\}$.

Let $\times M$ denote the direct product of all members of a set $M$ of matrices ($\times \emptyset = t_1$ and $\times \{A\} = A$). Put $\times^* K = \{\times M : M \subseteq K\}$. Clearly, $|\times^* K| = 2^n$ provided $|K| = n$.

**Theorem 2.** For each $F \subseteq F_p, \times^* \{2^f : f \in F\}$ is a closed semantics for $\models_F$.

The proof of Theorem 2 which generalizes the results from [3] is essentially based on the fact that $\to, \leftrightarrow, \leftarrow, \uparrow$ are independent in the sense of [1] and that the variety $V$ generated by the grupoids $(2, \to), (2, \leftrightarrow), (2, \leftarrow), (2, \uparrow)$ is strongly irregular, i.e. there is a term $\sigma(x, y)$ such that in $V$ holds the equation $\sigma(x, y) = x$ ([2, Example 1.7]).

The maximality degree of $\models_F$ strongly grows with $|F|$ but essentially depends also on the composition of $F$. E.g., for $|F| = 2$ it is $\leq 10$ and this bound is realized for $\{\uparrow\}$ (the dual of $\uparrow$) as easily follows from Theorem 2. On the other hand in many cases of $F := \{f, g\}, \models_2$ and $\models_3$ are the only proper nontrivial strengthenings of $\models_F$. An example is $F := \{\to, \leftrightarrow\}$. In this case $\{\to, \leftrightarrow\}$ is already closed because $2^\to \times 2^\leftrightarrow \equiv \{2^\to, 2^\leftrightarrow\}$. Since
each $\vdash \supseteq \vdash_F$ has tautologies, $\models_{2\rightarrow}$ and $\models_{2\rightarrow}$ are indeed the only proper nontrivial strengthenings of $\vdash_F$. Call $F \subseteq F_P$ ($|F| \geq 2$) nice whenever \{2$^f$ : $f \in F$\} is already closed, as in the last example. For a nice $F$, the $f \in F$ have a maximum of common rules, or, the calculus $\vdash_F$ is ambiguous to minimal extend. In particular, the only maximal strengthenings of $\vdash_F$ are the $\models_f$ for $f \in F$. From Theorem 2 it easily follows that $F$ is nice if and only if $F$ consists of some or all of the familiar connectives $\land, \lor, \rightarrow, \leftrightarrow$ and $\leftarrow$ which is essentially the same as $\rightarrow$). E.g., for $F_1 = \{\land, \lor, \rightarrow\}$, the favoured system of binary connectives, $\supseteq F_1$ has 7 proper nontrivial strengthenings only. Consider $F_2 = \{\land, +, \rightarrow\}$, i.e. “or” is replaced by “either-or”, $\supseteq F_2$ has nearly twice as many strengthenings as has $\vdash F_1$ which might explain to some extent the preference of $F_1$.

References


