CLASSICAL SUBTHEORIES AND INTUITIONISM

By $S$ we shall denote the set of all formulas in the language \{$\neg$, $\&$, $\lor$, $\Rightarrow$\} and by $C$ the classical consequence over $S$. The set of all theories we denote as $\text{Th}$. $\text{Th}_0$ is the set of all theories $T$ such that $T = C(\alpha)$, for some $\alpha \in S$. By $\text{Th}_1$, we denote the set $\text{Th}\setminus\text{Th}_0$. The set of all complete theories we denote as $\text{Cpl}$. For a given $T \in \text{Th}$ let $L_T = \{Y \subseteq T : Y \in \text{Th}\}$. It is obvious that for every $T \in \text{Th}$ the system $< L_T, \subseteq >$ is a lattice. It is evident that in this lattice $X \cup Y = C(X \cup Y)$ and $X \cap Y = X \cap Y$, for any $X, Y \in L_T$.

In [1] and [2] Dzik has proved among others that the lattice $< L_S, \subseteq >$ is an implicative lattice and its content (i.e. the set of all formulas in the language \{$\lor$, $\&$, $\Rightarrow$, $\neg$\} which are valid in the lattice $< L_S, \cup, \cap, \Rightarrow, \rightarrow >$) is equal to the set $\text{INT}$ of all formulas provable in the intuitionistic calculus. Here we will prove the following two theorems:

**Theorem 1.** If $T \in \text{Th}\setminus\{C(\emptyset)\}$, then $< L_T, \subseteq >$ is an implicative lattice and its content it equal to $\text{INT}$.

**Theorem 2.** Let $T_0 \in \text{Th}_0\setminus\{C(\emptyset)\}$ and $T_1 \in \text{Th}_1$. Then for every $T \in \text{Th}$ the lattice $< L_T, \subseteq >$ is isomorphic with one of the following lattices:

\[
< L_{C(\emptyset)}, \subseteq >, \quad < L_{T_0}, \subseteq >, \quad < L_{T_1}, \subseteq >.
\]

We define the function $J : \mathcal{P}(S) \longrightarrow \mathcal{P}(\text{Cpl})$ by:

\[J(X) = \{Z \in \text{Cpl} : X \not\subseteq Z\}, \text{ for any } X \subseteq S.\]

**Lemma 1.** (cf. [1], proof of Theorem 27).
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(a) \( J(C(X)) = J(X) \), for any \( X \subseteq S \)
(b) \( J(X) = \bigcup \{ J(\alpha) : \alpha \in X \} \), for any \( X \subseteq S \)
(c) \( J(\neg \alpha) = \text{Cpl} \setminus J(\alpha) \), for any \( \alpha \in S \)
(d) \( J(\alpha \& \beta) = J(\alpha) \cup J(\beta) \), for any \( \alpha, \beta \in S \)
(e) \( J(X) = \emptyset \) iff \( X \subseteq \text{Cpl}(\emptyset) \), for any \( X \subseteq S \)
(f) if \( \emptyset \neq R \subseteq \mathcal{P}(S) \), then \( J(\bigcup R) = \bigcup \{ J(X) : X \in R \} \)
(g) \( J(X) \cap J(Y) = J(C(X) \cap C(Y)) \), for any \( X, Y \subseteq S \)
(h) if \( T_1, T_2 \in \text{Th} \), then \( T_1 \subseteq T_2 \) iff \( J(T_1) \subseteq (T_2) \).

Note that from (a), (b) and (c) of the above Lemma it follows that the function \( J \) is an information function in the sense of [5]. Observe that by virtue of (e), (c), (f) and (g) of Lemma 1 the family \( T = \{ J(X) : X \subseteq S \} \) is a topology in the set \( \text{Cpl} \). By \( W \) we denote the topological space \( < \text{Cpl}, T > \). If \( Z \subseteq \text{Cpl} \) and \( T_Z = \{ Z \cap J(X) : X \subseteq S \} \), then naturally the system \( W_Z = < Z, T_Z > \) is a subspace of \( W \).

**LEMMA 2.** (cf [1], Theorem 27). Let \( T \in \text{Th} \). Then the lattice \( < LT, \subseteq > \) is isomorphic with the lattice \( < T_{J(T)}, \subseteq > \).

**PROOF.** Let \( T \in \text{Th} \). Let us consider the mapping \( H : LT \rightarrow T_{J(T)} \) given by \( H(Y) = J(Y) \), for every \( Y \in LT \). It is evident that \( H \) is an injection. We shall prove that it is a surjection. Let \( Z \in T_{J(T)} \), i.e. \( Z \) is an open set in the space \( W_{J(T)} \). Since \( J(T) \) is an open set in \( W \), then \( Z \) is also an open set in \( W \). From the definition of the space \( W \) there is a set \( X \subseteq S \) such that \( J(X) = Z \). By Lemma 1 (a), (h) we conclude that \( C(X) \in LT \) and \( H(C(X)) = Z \). From Lemma 1 (h) it follows that \( H \) is an isomorphism.

**LEMMA 3.** \( W \) is homeomorphic with the Cantor space.

**PROOF.** From Lemma 1 (b), (c) it follows that the set \( \{ J(\alpha) : a \in S \} \) is a basis of the space \( W \) and it consists of closed-open sets. Hence \( W \) is a zero-dimensional space. We easily prove that \( W \) is a regular space with a countable basis, so it is a metric space. Besides \( W \) is dense-in-itself and compact. It is well-known that every zero-dimensional metric, dense-in-itself, compact space is homeomorphic with the Cantor space. \( \square \)

We note that the set \( B_J = \{ J(\alpha) : \alpha \in S \} \) is a Boolean algebra of all closed-open subsets of the Cantor space (see Lemma 7 (a)). Note also that \( B_J \) is isomorphic with the Lindenbaum-Tarski algebra \( S/C(\emptyset) \).
Connections between Boolean algebras and the Cantor space are considered in [4].

**Lemma 4.** If $T \in \text{Th}\{C(\emptyset)\}$, then the lattice $\langle T_J(T), \subseteq \rangle$ is implicative and its content is equal to $\text{INT}$.  

**Proof.** Let $T \in \text{Th}\{C(\emptyset)\}$. Then $W_{J(T)}$ is a non-empty, open subspace of $W$. So $W_{J(T)}$ is a dense-in-itself, metric space. The lattice $\langle T_J(T), \subseteq \rangle$ of all open sets of $W$ is implicative and, according to the well-known result of McKinsey and Tarski (see [3], chapter IX, Theorem 3.2), its content is equal to $\text{INT}$. □

**Proof of Theorem 1** follows from Lemmas 2 and 4.

Now we are going to prove Theorem 2.

**Lemma 5.** Any two non-empty, closed-open subsets of the Cantor space are homeomorphic. Any two open, but not closed subsets of the Cantor space are homeomorphic. □

The next Lemma is an easy consequence of Lemmas 5 and 3:

**Lemma 6.** If $Z_1, Z_2 \subseteq \text{Cpl}$ are both non-empty, closed-open sets in $W$ or they are both open, but not closed sets in $W$ then the lattices $\langle T_{Z_1}, \subseteq \rangle$ and $\langle T_{Z_2}, \subseteq \rangle$ are isomorphic. □

**Lemma 7.** Let $Z \subseteq \text{Cpl}$ be an open set in $W$ and let $T \in \text{Th}$ be such that $J(T) = Z$. Then

(a) $Z$ is closed-open iff $T \in \text{Th}_0$

(b) $Z$ is open, but not closed iff $T \in \text{Th}_1$. □

**Proof of Theorem 2.** Let $T_0 \in \text{Th}_0\{C(\emptyset)\}$ and $T_1 \in \text{Th}_1$. Take up an arbitrary $T \in \text{Th}$. If $T \in \text{Th}_0\{C(\emptyset)\}$, then $J(T)$ is a non-empty, closed-open set in $W$ (see Lemma 1 (c) and Lemma 7). Hence by Lemmas 6 and 2 the lattice $\langle L_T, \subseteq \rangle$ is isomorphic with the lattice $\langle L_{T_0}, \subseteq \rangle$. We analogously prove that if $T \in \text{Th}_1$, then $\langle L_T, \subseteq \rangle$ is isomorphic with $\langle L_{T_1}, \subseteq \rangle$. □
References


