ON A CLOSURE OF A SET
OF CLASSICALLY ESSENTIAL FORMULAS
AND ON RELEVANT IMPLICATION

A generalization of the notion of essential occurrence of propositional
variable in a formula introduced in [1] is formulated in [5]. The given work
presents a result obtained in the process of investigating relevant logics
with the above generalization. This result which connects the implicative
fragment of a well-known relevant system $R$ with a set (closed with respect
to the rules of substitution and detachment) of all classically essential for-
mulas of implicative fragment of classical propositional logic the descriptive
length of which is less than 6 allows a new decision of the problem con-
cerning the nature of relevant implication.

The implicative fragment $R \supset$ by Anderson and Belnap (see, for ex-
ample, [2]) is formulated in the propositional language $L \supset$ the alphabet of
which contains only propositional variables $p_1, p_2, p_3, \ldots, p_n$, parentheses
and one logical connective-implication $\supset$.

$R \supset$ is the least set of formulas of $L \supset$ which contains the formulas:
(1) $p_1 \supset p_1$
(2) $(p_1 \supset p_2) \supset ((p_2 \supset p_3) \supset (p_1 \supset p_3))$
(3) $(p_1 \supset (p_2 \supset p_3)) \supset (p_2 \supset (p_1 \supset p_3))$
(4) $(p_1 \supset (p_1 \supset p_2)) \supset (p_1 \supset p_2)$
and is closed under the rules of substitution and detachment.

Following [5] in the definitions given below $L$ is a propositional lan-
guage (in the general case not necessarily purely implicative), $\Gamma$ is a subset
of the set of all formulas in $L$, $A \in \Gamma, W(A)$ is a set of all propositional
variables occurring in $A, p \in W(A)$.

We shall call a given occurrence of $p$ in $A$ $\Gamma$-essential if for any proposi-
tional variable $q$ of the language $L$ not belonging to $W(A)$, the result of
substitution of \( p \) in \( A \) for \( q \) does not belong to \( \Gamma \).

\( p \) is called \( \Gamma \)-essential in \( A \) if any occurrence of \( p \) in \( A \) is \( \Gamma \)-essential occurrence of \( p \) in \( A \).

A formula \( A \) is called \( \Gamma \)-essential if any propositional variable occurring in \( A \) is a \( \Gamma \)-essential variable in \( A \).

The number of occurrences of propositional variables in \( A \) will be called the descriptive length of the formula \( A \) in language \( L_\supset \) (symbolically \( d(A) \)), that is

\[
d(A) = \begin{cases} 
1 & \text{if } A \text{ is a propositional variable} \\
1 + d(B) + d(C) & \text{if } A \text{ is of the form } B \supset C 
\end{cases}
\]

Let \( C_\supset \) be a set of all formulas in \( L_\supset \) being classical tautologies (valid formulas) in a propositional language and let \( C_\supset^* \) be the closure with respect to the rules of substitution and detachment of the set of all \( C_\supset \)-essential formulas with descriptive length less or equal to 6.

**Theorem.** \( R_\supset = C_\supset^* \).

The proof of the theorem is divided into two parts.

I. \( R_\supset \subseteq C_\supset^* \).

In fact, it is an easy work to check that each of the formulas (1)–(4) is a \( C_\supset \)-valued formula with descriptive length less or equal to 6 and thus taking into account that \( R_\supset \) and \( C_\supset^* \) are closed under the same rules we have

\[ R_\supset \subseteq C_\supset^* \]

II. \( C_\supset^* \subseteq R_\supset \).

We shall need the following lemmas:

**Lemma 1.** Let \( \Pi \) be a finite set of propositional variables of the language \( L_\supset \). The set \( M_\Pi^n = \{ A \mid A \text{ is a formula in } L_\supset, d(A) = n, W(A) \subseteq \Pi \} \) is finite.

Lemma 1 is easily proved by induction on \( n \) giving the constructive method for generating \( M_\Pi^n \) for any fixed \( \Pi \) and any \( n (= 1, 2, 3, \ldots) \).

**Lemma 2.** Assume that \( n = 1, 2, 3, \ldots \) and \( \Pi_n \) is the set of all propositional
variables \(p_i\) of the language \(L_\succ\) such that \(1 \leq i \leq n\). If \(\{A \mid A \text{ is } C_\succ\text{-essential formula, } d(A) \leq n, W(A) \subseteq \Pi_n\} \subseteq R_\succ\) then \(\{A \mid A \text{ is } C_\succ\text{-essential formula, } d(A) \leq n\} \subseteq R_\succ\).

We shall prove Lemma 2 using the reductio ad absurdum method. Assume that in \(C_\succ\) is an essential formula \(A\) of descriptive length \(n\) such that \(A \not\in R_\succ\). It is evident that \(S_{p_1, \ldots, p_k} A\) (the result of simultaneous substitution of \(p_1, \ldots, p_k\) for \(p_i_1, \ldots, p_i_k\), respectively, where \(p_i_1, \ldots, p_i_k\) are all relatively different variables occurring in \(A\)) is a \(C_\succ\text{-essential formula with descriptive length } n\) and \(W(S_{p_1, \ldots, p_k} A) \subseteq \Pi_n\).

Hence it follows from the condition of Lemma 2 that

\[ S_{p_1, \ldots, p_k} A \in R_\succ \]

But then \(S_{p_1, \ldots, p_k} S_{p_1, \ldots, p_k} A \in R_\succ\) (\(R_\succ\) is closed under the rule of substitution), i.e. \(A \in R_\succ\). Lemma 2 is proved.

Now consider the set \(M = \{A \mid A \text{ is a formula in } L_\succ, d(A) \leq 6, W(A) \subseteq \Pi_6\}\), where \(\Pi_6\) is defined like in Lemma 2. \(M\) is a finite set since \(M = \bigcup_{i=1}^6 M_{\Pi_6}\) is a union of six finite (according to Lemma 1) sets.

In virtue of the constructive character of Lemma 1 the set \(M\) may be put by listing all elements belonging to it. The procedure of extracting all \(C_\succ\text{-essential formulas of this list and checking that every of the formulas belongs to } R_\succ\) is purely mechanical because the set of all \(C_\succ\text{-essential formulas is evidently decidable and } R_\succ\) is also decidable as proved in [4].

Here we omit the tiresome details of ‘extraction’ and ‘checking’.

Thus \(M \subseteq R_\succ\). Hence it follows (according to Lemma 2) that \(\{A \mid A \text{ is } C_\succ\text{-essential formula, } d(A) \leq 6\} \subseteq R_\succ\). Taking into account that \(C_\succ^*\) and \(R_\succ\) are closed under the same rules we obtain \(C_\succ^* \subseteq R_\succ\).

In conclusion it should be noted that \(C_\succ^*\) is a proper subset of \(C_\succ^\ast\), which is the closure under the rules of substitution and detachment of the set of all \(C_\succ\text{-essential formulas.}

Here we have an example of \(C_\succ\text{-essential formula which does not belong to } C_\succ^*\)

\[ (((p_1 \supset p_2) \supset p_3) \supset p_4) \supset (((p_1 \supset p_4) \supset ((p_2 \supset p_4) \supset p_2)) \supset p_2)\]
The author cannot answer the following questions which naturally suggest themselves in the context of the article:

What are “good” axiomatizations of $C^c$?

Is $C^c$ a relevant logic in the sense of [3]?

References


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